

Nonabelian (p, p) classes

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Abstract

In this paper we generalize to the non-abelian context a classical theorem of Griffiths which studies the behavior of the (p, q) -components of a horizontal section in a variation of Hodge structures over a smooth projective variety.

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1 Introduction

In this paper we generalize to the non-abelian context a classical theorem of Griffiths [Griffiths70, Theorem 7.1] which studies the behavior of the (p, q) -components of a horizontal section in a variation of Hodge structures over some smooth projective variety. We begin by recalling the abelian story.

1.1 Abelian (p, p) classes

One of the geometric motivations of the question addressed by Griffiths theorem of the (p, q) classes comes from the Hodge conjecture. Recall that for any complex smooth projective variety X the Hodge theorem provides a canonical isomorphism (see Section 3.1 for details):

$$\oplus_{p+q=w} H^p(X, \Omega_X^q) \cong H^w(X, \mathbb{C}).$$

The set \mathcal{W}^p consisting of classes of algebraic cycles of codimension p is contained in the locus of classes of type (p, p) , i.e we have an inclusion

$$\mathcal{W}^p \subset H^p(X, \Omega_X^p) \cap H^{2p}(X, \mathbb{Z}) \subset H^{2p}(X, \mathbb{C}).$$

The rational Hodge conjecture then says that the locus $\mathcal{W}^p \otimes \mathbb{Q}$ of motivic cohomology classes coincides with the locus $H^p(X, \Omega_X^p) \cap H^{2p}(X, \mathbb{Q})$ of Hodge classes.

It is therefore clear that understanding the Hodge conjecture requires having a good grasp on the locus of Hodge classes or more modestly on the the locus $H^p(X, \Omega_X^p) \cap H^{2p}(X, \mathbb{C})$ of (p, p) -classes.

One way to gauge the plausibility of the Hodge conjecture is by deforming X . A beautiful geometric method for producing potential counter-examples to the Hodge conjecture was proposed by André Weil [Weil79]. It is based on the observation that subvarieties tend to be much more rigid than the cohomology classes they represent. Weil suggests that one looks for a smooth variety X and a subvariety $Y \subset X$ with a cohomology class $\alpha = \text{PD}[Y] \in H^p(X, \Omega_X^p) \cap H^{2p}(X, \mathbb{Q})$ such that:

- the pair (X, α) deforms to a pair $(f : \mathfrak{X} \rightarrow S, a : S \rightarrow R^{2p}f_*\mathbb{Q})$ with $a \in \Gamma(S, R^p f_* \Omega_{\mathfrak{X}/S}^p)$ and $a(o) = \alpha$ for some $o \in S$.
- the deformations of Y in \mathfrak{X} are obstructed.

In this setup the classes $\{a(s)\}_{s \neq o}$ are candidates for counter-examples to the Hodge conjecture.

Remark 1.1 In fact Weil even proposes an explicit construction of pairs $X \subset Y$ coming from correspondences on abelian varieties of CM-type. However C. Schoen had shown [Schoen88] that the deformed classes in Weil's examples are also represented by algebraic cycles.

Weil's observation shows that it is important to find ways of deforming cohomology classes so that properties like Hodge type and rationality are preserved.

On the other hand, there is an intrinsic way to deform cohomology classes - a parallel transport via the Gauss-Manin connection. Indeed given a smooth projective morphism $f : X \rightarrow S$ between smooth varieties the vector bundles $(R^m f_* \mathbb{C}_X) \otimes_{\mathbb{C}} \mathcal{O}_S$ have a natural integrable connection - *the Gauss-Manin connection* - coming from the inclusion

$$R^m f_* \mathbb{Z}_X / \text{torsion} \subset R^m f_* \mathbb{C}_X$$

(see Section 3.1 for details). Thus, given a cohomology class $\alpha \in H^{2p}(X_o, \mathbb{C})$ we have a unique horizontal extension $a \in \Gamma(U, R^m f_* \mathbb{C}_X)$ of α over some simply-connected analytic neighborhood $o \in U \subset S$. If the α we started with was a rational cohomology class, then by the definition of the Gauss-Manin connection we will have $a \in \Gamma(U, R^m f_* \mathbb{Q}_X)$. So, as far as deformations are concerned, the rationality of a cohomology class does not pose any major obstructions. The next question to ask is if the Gauss-Manin connection preserves the property of being of Hodge type (p, p) . This is a non-trivial question since the images of $R^p f_* \Omega_X^q$ in $(R^{p+q} f_* \mathbb{C}_X) \otimes_{\mathbb{C}} \mathcal{O}_S$ are not algebraic subsheaves and are not preserved by the Gauss-Manin connection.

The answer to this question is given by the following theorem (see Definition 3.2 for the definition of a variation of Hodge structures):

Theorem([Griffiths70, Theorem 7.1]) *Let S be a smooth complex projective variety and let $(V_{\mathbb{Z}} \subset V, \nabla, F^{\bullet})$ be a variation of Hodge structures on S . Then for any horizontal global section a of V , the Hodge (p, q) -components of a are also horizontal.*

In particular if a is of pure Hodge type (p, q) at some point, then a is of pure Hodge type (p, q) everywhere.

As a consequence one has the following immediate

Corollary *Let $f : X \rightarrow S$ be a smooth morphism of smooth projective varieties. Let $a \in \Gamma(S, R^{2p} f_* \mathbb{C}_X \otimes \mathcal{O}_S)$ an algebraic global section which is horizontal with respect to the Gauss-Manin connection and such that $a(o)$ is of Hodge type (p, p) for some point $o \in S$. Then $a(s)$ is of Hodge type (p, p) for every s .*

It is exactly this corollary that we wish to generalize to the nonabelian situation. Before we explain what this means it is instructive to examine the existing generalizations of Griffiths theorem in the abelian situation.

The first is Deligne's theorem of the fixed part. Given a morphism $f : X \rightarrow S$ as in the previous corollary consider the sub-local system $\mathbb{V} \subset R^w f_* \mathbb{C}_X$ spanned by the global

horizontal sections. Then Griffiths theorem on the (p, q) -classes is equivalent to saying that the C^∞ decomposition of $(R^w f_* \mathbb{C}_X) \otimes \mathcal{O}_S$ into (p, q) -pieces induces a horizontal (p, q) -decomposition of \mathbb{V} , i.e. that \mathbb{V} is a sub variation of Hodge structures. One can ask if this is always the case and this what Deligne's theorem answers¹:

Theorem([Deligne72]) *Let $f : X \rightarrow S$ be a smooth projective morphism to a quasi-projective S . Let G be the Zariski closure of the monodromy group of $R^w f_* \mathbb{C}_X$ and let V be a representation of G which is defined over \mathbb{Q} . Then*

- (1) (see [Deligne72, Section 4.2]) *G is a complex reductive group.*
- (2) (see [Deligne72, Section 4.1]) *The isotypic component $\mathbb{V} \subset R^w f_* \mathbb{C}_X$ of V is a \mathbb{Q} -sub-variation of Hodge structures.*

(Recall that the isotypic component of V is by definition the maximal sub-local system that corresponds to a direct sum of copies of V .)

Deligne's theorem of the fixed part was generalized further to the case of general variations of Hodge structures (i.e. variations not necessarily of geometric origin) by W. Schmid:

Theorem ([Schmid73, Theorem 7.22]) *Let S be a smooth complex manifold that can be embedded as a Zariski open subset in a compact analytic space. Let G be the Zariski closure of the monodromy group of $R^w f_* \mathbb{C}_X$ and let V be a representation of G which is defined over \mathbb{Q} . Then the isotypic component $\mathbb{V} \subset R^w f_* \mathbb{C}_X$ of V is a \mathbb{Q} -sub-variation of Hodge structures.*

Clearly the two theorems of Deligne and Schmid quoted above specialize to Griffiths theorem of the (p, q) -classes after taking S -projective and V - the trivial one dimensional G -module.

In a slightly different direction one can forget about the integral structure of $R^w f_* \mathbb{C}_X$ and view it just as a complex variation of Hodge structures (see Definition 3.4). In other words we look at the data $(V, \nabla, F^\bullet, \overline{F}^\bullet)$ where $V := (R^w f_* \mathbb{C}_X) \otimes \mathcal{O}_S$, ∇ is the holomorphic part of the Gauss-Manin connection, $F^p := \bigoplus_{r \geq q} R^q f_* \Omega_{X/S}^r$ and \overline{F}^\bullet is the complex conjugate of F^\bullet . Since ∇ integrable and satisfies the Griffiths transversality condition $\nabla(F^p) \subset F^{p-1} \otimes \Omega_S^1$ we can form the associated graded $(E, \theta) = (\text{gr}_{F^\bullet}(V), \text{gr}_{F^\bullet}(\nabla))$. That is $E = \bigoplus_{p+q=w} R^q f_* \Omega_{X/S}^p$ and $\theta : E \rightarrow E \otimes \Omega_S^1$ is a morphism of coherent sheaves satisfying $\theta \wedge \theta = 0$. Such pairs (E, θ) are called *Higgs bundles* and their appearance in the context of variations of Hodge structures is the starting point of the non-abelian Hodge theory of Simpson that we will be concerned with.

Note that if an algebraic section $a : S \rightarrow V$ is horizontal with respect to the non-abelian Gauss-Manin connection, then it will necessarily lie in the kernel of θ . Actually something

¹This is not exactly the way Deligne states his theorem but the method of proof gives this slightly stronger result.

more is true. If we put \mathcal{H} for the C^∞ bundle underlying both V and E , then for any C^∞ section a of \mathcal{H} we have (see e.g. the proof of [Schmid73, Theorem 7.22]):

$$(1.1.1) \quad \left(\begin{array}{l} a \text{ is a holomorphic section of} \\ V \text{ satisfying } \nabla a = 0 \end{array} \right) \iff \left(\begin{array}{l} a \text{ is a holomorphic section of} \\ E \text{ satisfying } \theta a = 0 \end{array} \right)$$

In fact (1.1.1) implies Griffiths theorem of the (p, q) -classes. The main point is to observe that the connection $D = \bar{\partial}_E + \nabla + \theta$ is a Hermitian connection on \mathcal{H} corresponding to the Hermitian metric on \mathcal{H} that one obtains by flipping the signs of the polarization on V (see Definition 3.4) on the appropriate (p, q) -pieces. In particular the decomposition of \mathcal{H} into (p, q) -pieces is D -horizontal and hence if $a \in C^\infty(S, \mathcal{H})$ is a solution to the initial value problem $Da = 0$, $a(o) = a_0 \in (R^q f_* \Omega_{X/S}^q)$, then $a \in C^\infty(S, R^q f_* \Omega_{X/S}^q)$.

Now $\nabla a = 0$ combined with (1.1.1) implies that $Da = (\bar{\partial}_E + \nabla + \theta)a = \bar{\partial}_E a + \nabla a + \theta a = 0$ and hence one gets the theorem of the (p, q) classes.

The statement (1.1.1) admits a far reaching generalization which follows from the higher order Kähler identities in non-abelian Hodge theory. More precisely recall that for a smooth projective variety S the non-abelian Hodge theorem of Corlette-Simpson provides a correspondence between complex reductive local systems on S of rank n and poli-stable rank n Higgs bundles on S with vanishing rational Chern classes (see Section 3.1 for details). Now one has the following formality theorem:

Theorem ([Simpson92, Lemma 2.2]) *Let S be a smooth projective variety and let (V, ∇) be a complex local system with reductive monodromy. Let (E, θ) be the corresponding poli-stable Higgs bundle. Then there is a canonical isomorphism*

$$H^\bullet(S, (V, \nabla)) \cong H^\bullet(S, (E, \theta)).$$

This general theorem specializes to (1.1.1) when we take (V, ∇) to be a complex variation of Hodge structures and use cohomology of degree zero.

In fact as we will show in Proposition 5.6 the proof of the non-abelian version of the theorem of the (p, p) classes follows formally from the Simpson formality theorem.

1.2 Statement of the main theorem

In order to generalize the previous discussion to the non-abelian setting we have to explain what the non-abelian versions of the (p, p) -classes and the Gauss-Manin connection are. Most of the relevant concepts were discovered and extensively studied by Simpson (see Section 3.1 for a short description and the precise references to the original works).

We will discuss only the case of the first non-abelian cohomology since it is the most geometric one. All essential features of the problem carry over to the case of the higher

degree cohomology [Simpson99], [Hirschowitz-Simpson98] but this is beyond the scope of the present paper.

The right way (see Section 3.1 and references therein) to view the first de Rham cohomology of a smooth projective variety Y is as the moduli (stack) $\mathcal{M}_{\text{DR}}(Y, n)$ of rank n local systems on Y . The non-abelian (p, p) classes in $\mathcal{M}_{\text{DR}}(Y, n)$ can be defined as the set of fixed points for the action of certain Weil operators (generalizing the abelian ones) and were identified by Simpson [Simpson92, Lemma 4.1] as the local systems underlying complex variations of Hodge structures.

Next, for a smooth projective morphism $f : X \rightarrow S$ with connected fibers one looks at the family of de Rham cohomology along the fibers $\pi_{\text{DR}} : \mathcal{M}_{\text{DR}}(X/S, n) \rightarrow S$. Similarly to the case of variations of Hodge structures of geometric origin one can show that π_{DR} carries a natural structure of a local system of stacks (or a crystal) over S . Roughly speaking this means that locally over S the family π_{DR} is equipped with a canonical trivialization over any infinitesimal thickening of S or equivalently $\mathcal{M}_{\text{DR}}(X/S, n)$ is equipped with an action of the sheaf of differential operators \mathcal{D}_S on S .

The algebraic construction of this non-abelian connection was again invented by Simpson [Simpson95] who dubbed it the *non-abelian Gauss-Manin connection*. We discuss this construction at length in Section 4.1. To make things more explicit though here we will use the analytic definition of the non-abelian Gauss-Manin connection which is more down to earth.

Assume for simplicity that $f : X \rightarrow S$ has a section ξ . The geometric Riemann-Hilbert correspondence [Simpson95, Proposition 7.8] identifies the underlying analytic stack of $\mathcal{M}_{\text{DR}}(X/S, n)$ with the moduli stack $\mathcal{M}_{\text{B}}(X/S, n)$ of n -dimensional complex representations of the fundamental groups of the fibers of f . Furthermore since f has a section one can use the covering homotopy theorem in order to define a monodromy action of $\pi_1(S, o)$ on the fundamental group of the fiber X_o . By construction of the monodromy action the elements of $\pi_1(S, o)$ act by group automorphisms of $\pi_1(X_o, \xi(o))$ and so one can compose them with representations $\pi_1(X_o, \xi(o)) \rightarrow \text{GL}_n(\mathbb{C})$ to obtain an action of $\pi_1(S, o)$ on $\mathcal{M}_{\text{DR}}(X_o, n)$. This action is precisely the analytic version of the non-abelian Gauss-Manin connection. In particular an algebraic section $a : S \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$ will be horizontal with respect to the non-abelian Gauss-Manin connection if and only if $a(o)$ is fixed under $\pi_1(S, o)$ (in the sense of group actions on stacks).

With all of this said we are now ready to state our main theorem

Theorem A *Let $f : X \rightarrow S$ be a smooth projective morphism with connected fibers. Assume that S is projective and let $a : S \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$ be an algebraic section of π_{DR} which is horizontal with respect to the non-abelian Gauss-Manin connection. If there exists a point $o \in S$ so that $a(o)$ underlies a complex variation of Hodge structures, then $a(s)$ underlies a complex variation of Hodge structures for all $s \in S$.*

Remark 1.2 It is very natural to ask if the result of Theorem A will hold for a quasi-projective base scheme S . Due to the lack of the necessary analysis this seems to be out of reach at the moment. See however Corollary 5.8 for a slightly weaker statement.

Geometrically the proof of this theorem boils down to the statement that if (F, ∇) is a global local system on X whose restriction on X_o underlies a complex variation of Hodge structures, then the restriction of (F, ∇) to any X_s underlies a complex variation of Hodge structures. This fact is the content of Proposition 5.6. As mentioned before it follows ultimately from Simpson's formality theorem [Simpson92, Lemma 2.2]. To illustrate what is going on we present a direct argument in the simple (abelian) case of rank one local systems.

Example 1.3 Let X and S be smooth and projective varieties and let $f : X \rightarrow S$ be a smooth morphism with connected fibers. Let (F, ∇) be a rank one complex local system on X such that $(F_o, \nabla_o) := (F, \nabla)|_{X_o}$ underlies a complex variation of Hodge structures. Recall that a rank one local system underlies a complex variation of Hodge structures if and only if it is unitary i.e. if and only if the connection preserves a hermitian metric on the bundle. In terms of representations of the fundamental group this just means that the character χ corresponding to (F_o, ∇_o) is a character $\chi : \pi_1(X_o) \rightarrow U(1) \subset \mathbb{C}^\times$. Since the monodromy action of $\pi_1(S, o)$ on $\mathcal{M}_{\text{DR}}(X_o, n)$ is just the composition of χ with automorphisms of $\pi_1(X_o)$ it is clear that for all $\gamma \in \pi_1(S, o)$ the characters χ and $\gamma^*\chi$ will have the same image, i.e. the property of χ being a CVHS is preserved.

This argument however is completely misleading since in this simple case the property of (F_s, ∇_s) underlying a CVHS is entirely topological (i.e. the monodromy of (F_s, ∇_s) should be compact) and does not depend on the complex structure on X_s . This of course is not true in general so the previous argument cannot generalize to a proof of Theorem A. Therefore it will be more helpful to have an algebraic argument that uses the geometric structure on X and S . Such an argument is not hard to find. Indeed - since F has a holomorphic integrable connection it follows that $c_1(F) = 0 \in H^1(X, \Omega_X^1)$ and so we can find [Griffiths-Harris94, Section 1.2] a flat unitary connection D on F . Let $\theta = D^{1,0} - \nabla$. Then α is a holomorphic one form on X and the condition that (F_o, ∇_o) is unitary is equivalent to saying that the image of α under the restriction map $r_o : \Omega_X^1 \rightarrow \Omega_{X_o}^1$ is zero. From this we would like to deduce that the image of α under any restriction map $r_s : \Omega_X^1 \rightarrow \Omega_{X_s}^1$ is zero.

More invariantly put Ω_f^1 for the sheaf of holomorphic one forms along the fibers of f and let $r : \Omega_X^1 \rightarrow \Omega_f^1$ be the obvious surjection. Then the global one form α maps to a section $r(\alpha) = H^0(X, \Omega_f^1) = H^0(S, f_*\Omega_f^1)$ which vanishes at o . We want to show that $r(\alpha) = 0 \in H^0(S, f_*\Omega_f^1)$. For this consider the weight one variation of Hodge structures $((R^1f_*\mathbb{C}_X) \otimes \mathcal{O}_S, \text{GM})$ with GM being the Gauss-Manin connection. By Griffiths infinitesimal period relations [Griffiths69, Section 9] the Higgs bundle corresponding to this variation is

$$(E, \theta) := (\text{gr}_{F^\bullet}(R^1f_*\mathbb{C}_X) \otimes \mathcal{O}_S, \text{gr}_{F^\bullet} \text{GM}) = (f_*\Omega_f^1 \oplus R^1f_*\mathcal{O}_X, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix})$$

where $c : f_*\Omega_f^1 \rightarrow R^1f_*\mathcal{O}_X \otimes \Omega_S^1$ is the cup product with the Kodaira-Spencer class $\kappa_{X/S} \in R^1f_*T_f \otimes \Omega_S^1$ of f .

Recall next that $\kappa_{X/S}$ is defined as the first edge homomorphism for the push-forward long exact sequence of the following short exact sequence of sheaves on X

$$0 \rightarrow T_f \rightarrow T_X \rightarrow f^*T_S \rightarrow 0.$$

Therefore c is just the first edge homomorphism of the push-forward long exact sequence for

$$0 \rightarrow f^*\Omega_S^1 \rightarrow \Omega_X^1 \xrightarrow{r} \Omega_f^1 \rightarrow 0$$

and so $\ker(c) = \text{im}(r)$. This shows that $r(\alpha)$ is a holomorphic section of E which is annihilated by θ and so by (1.1.1) $r(\alpha)$ can be interpreted as an algebraic section of $(R^1f_*\mathbb{C}_X) \otimes \mathcal{O}_S$ which is GM horizontal. Hence if $r(\alpha)$ vanishes at one point it must vanish everywhere.

Here is a brief outline of the content of this paper. In Section 2 we recall the notions of a local system and a \mathcal{D} -module and a local systems of schemes and a \mathcal{D} -scheme. In Section 3 with the hope of making the paper more readable we review the necessary background from non-abelian Hodge theory. The main part of the paper begins in Section 4 where we interpret the non-abelian Gauss-Manin connection in terms of deformation theory and find an explicit condition for a section to be horizontal. Whenever possible we have adopted the Čech point of view hoping to make the arguments more transparent. Section 4.2 discusses the case of variations of non-geometric origin and possible directions of generalizing Theorem A. In Section 5 we prove the main theorem first in the case of curves and then in general. For the convenience of the reader we have collected in an Appendix some well known fact about algebraic stacks that are used throughout the paper.

The proof of the theorem of the non-abelian (p, p) -classes presented here was finished in late 1996. Since then we have reported on this work on several occasions explaining details of the proof. In particular we have given lectures on the subject during the Warwick Symposium in the Summer of 1997, the Oberwolfach Complex Geometry Workshop in September 1997, and the ICM Satellite Conference in Essen in August 1998. We apologize for the delay in writing this result up.

During the preparation of this paper C. Simpson informed us that J. Jost and K. Zuo (who were both present at the Oberwolfach lecture of L.K.) have announced similar results.

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1.3 Notation and terminology

In this section we list the basic notions used throughout the paper and give page references for the place in the text where they are explained.

General notation

$\mathcal{E}(F)$ the Atiyah algebra of a coherent sheaf F . See also $\mathcal{D}_S^i(F)$. *page 12*

$e(F)$ the Atiyah class of a coherent sheaf F . See also $\kappa_{X/S}$ and $e(X/S)$. *page 12*

$e(X/S)$ the Atiyah class of a family of schemes $f : X \rightarrow S$. See also $e(F)$ and $\kappa_{X/S}$. *page 14*

$f : X \rightarrow S$ usually a smooth projective morphism between smooth varieties with connected fibers. *page 2*

$F(\text{or } E)$ a vector bundle. *page 12*

d^∇ an algebraic connection on a coherent sheaf F interpreted as a differential. See also ∇ . *page 12*

∇ an algebraic connection on a coherent sheaf F interpreted as a splitting of the Atiyah sequence. *page 12*

\mathcal{D}_X^\sim the centralizer of \mathcal{D}_f in \mathcal{D}_X . See also T_X^\sim . *page 15*

T_X^\sim the centralizer of the vertical tangent sheaf T_f with respect to the Lie bracket on T_X . *page 14*

$\text{Sch}_{\mathcal{D}}(S)$ the category of \mathcal{D}_S -schemes. *page 17*

\mathcal{D}_S the sheaf of algebraic differential operators acting on functions on S . *page 12*

$\mathcal{D}_S^i(F)$ the sheaf of algebraic differential operators of order $\leq i$ acting from F to F . *page 12*

$(S \times \dots \times S)^\wedge$ the complete formal neighborhood of the small diagonal in $S \times \dots \times S$. *page 16*

τ_X the isomorphism of de Rham and Dolbeault cohomology provided by the (abelian or nonabelian) Hodge theorem. *page 21*

$\kappa_{X/S}$ the Kodaira-Spencer class of a family of schemes $f : X \rightarrow S$. See also $e(F)$ and $e(X/S)$. *page 14*

S a smooth (usually quasi projective) scheme over \mathbb{C} . *page 11*

S_{DR} the de Rham formal groupoid corresponding to a smooth scheme S or sometimes the corresponding quotient stack of S . *page 17*

T_S the Zariski tangent sheaf of a scheme S . *page 11*

\mathcal{D}_f the sheaf of vertical differential operators for a morphism $f : X \rightarrow S$. See also T_f . *page 15*

T_f the sheaf of tangent vectors tangent to the fibers of a morphism $f : X \rightarrow S$. *page 14*

Categories

\mathcal{A} an abelian category. *page 30*

(Ab) the category of all abelian groups. *page 68*

BG the classifying groupoid of a group scheme G over S . *page 62*

$\text{Comm}_{\mathcal{D}}(S)$ the category of commutative \mathcal{D}_S -algebras. *page 15*

$\text{Comm}_{\mathcal{O}}(S)$ the category of unital commutative \mathcal{O} -algebras. *page 15*

$\mathcal{M}_{\mathcal{D}}^l(S)$ the category of all left \mathcal{D} -modules on a scheme S . *page 13*

(Grp) the 2-category of all groupoids. *page 62*

$\mathcal{M}_{\mathcal{O}}(S)$ the category $\mathcal{M}_{\mathcal{O}}(S)$ of quasi-coherent \mathcal{O}_S -modules on a scheme S . See also $\text{Qcoh}_{X/S}$. *page 13*

$\text{Qcoh}_{X/S}$ the groupoid of quasi coherent S -flat \mathcal{O}_X -modules. See also $\mathcal{M}_{\mathcal{O}}(S)$. *page 62*

(Sch/ S) the category of all schemes over S . *page 62*

(Set) the category of all sets. *page 63*

$\text{Vect}_{\mathbb{C}}$ the category of complex vector spaces. *page 13*

Hodge theoretic concepts

$\mathcal{V}_{\mathbb{Z}}^p$ the locus of (abelian) Hodge classes of codimension p on X . See also $\mathcal{V}_{\mathbb{C}}^p$. *page 22*

$\mathcal{V}_{\mathbb{C}}^p$ the locus of (abelian) (p, p) -classes on X . See also $\mathcal{V}_{\mathbb{Z}}^p$. *page 22*

ϕ_X the isomorphism of Betti and Dolbeault cohomology. *page 22*

$\mathbf{Hod}_{2p}(X)$ the Hodge group acting on the degree $2p$ cohomology of X . *page 23*

$H_{\text{Del}}^1(X, \text{GL}_n(\mathbb{C}))$ the Deligne moduli space of rank n twisted connections on X . *page 29*

$\mathcal{V}_{\mathbb{Z}}$ the locus of nonabelian Hodge classes on X . See also $\mathcal{V}_{\mathbb{C}}$. *page 24*

$H_{\text{Hod}}^1(X, \text{GL}_n(\mathbb{C}))$ the moduli space of rank n -lambda connections on X . *page 29*

$\mathcal{V}_{\mathbb{C}}$ the locus of nonabelian (p, p) -classes on X . See also $\mathcal{V}_{\mathbb{Z}}$. *page 24*

ψ_X the Riemann-Hilbert isomorphism of the nonabelian Betti and de Rham cohomology. *page 20*

Operations on complexes

$\tau_{\leq n}(\tau_{\geq n})$ the canonical truncations of a complex in an abelian category. See also $\sigma_{\leq n}(\sigma_{\geq n})$.
page 37

$\sigma_{\leq n}(\sigma_{\geq n})$ the stupid truncation of a complex in an abelian category. See also $\tau_{\leq n}(\tau_{\geq n})$.
page 30

Cohomology groups

$H_{\mathbf{B}}^{\bullet}(X, A)$ the Betti cohomology of X with coefficients in an abelian group. *page 19*

$H_{\mathbf{Dol}}^{\bullet}(X, A)$ the Dolbeault cohomology of X with coefficients in A , where A is an affine commutative algebraic group over \mathbb{C} . *page 20*

$H_{\mathbf{DR}}^{\bullet}(X, A)$ the de Rham cohomology of X with coefficients in A , where A is an affine commutative algebraic group over \mathbb{C} . *page 19*

$H_{\mathbf{B}}^1(X, G)$ the moduli space of semisimple representations $\pi_1(X) \rightarrow G$ into a complex reductive group G . *page 20*

$H_{\mathbf{DR}}^1(X, G)$ the moduli space of G local systems on X for a reductive algebraic group G .
page 20

Moduli stacks

$\mathcal{B}un(X/S, n)$ the relative stack of rank n vector bundles with vanishing rational Chern classes. See also $\mathcal{B}un^o(X/S, n)$. *page 52*

$\mathcal{B}un^o(X/S, n)$ the open substack of $\mathcal{B}un(X/S, n)$ over which the structure morphism $\pi : \mathcal{B}un(X/S, n) \rightarrow S$ is smooth. *page 52*

$\mathcal{M}_{\mathbf{Dol}}(X/S, n)$ the moduli stack of relative rank n Higgs bundles on X/S . *page 31*

$\mathcal{M}_{\mathbf{Dol}}^o(X/S, n)$ the open substack in $\mathcal{M}_{\mathbf{Dol}}(X/S, n)$ over which the structure morphism $\pi_{\mathbf{Dol}} : \mathcal{M}_{\mathbf{Dol}}(X/S, n) \rightarrow S$ is smooth. *page 33*

$\mathcal{M}_{\mathbf{DR}}(X/S, n)$ the moduli stack of relative rank n local systems on X/S . *page 31*

$\mathcal{M}_{\mathbf{DR}}^o(X/S, n)$ the open substack in $\mathcal{M}_{\mathbf{DR}}(X/S, n)$ over which the structure morphism $\pi_{\mathbf{DR}} : \mathcal{M}_{\mathbf{DR}}(X/S, n) \rightarrow S$ is smooth. *page 33*

2 Preliminaries on \mathcal{D} -varieties

Let S be a smooth scheme over the complex numbers. The main subject of our investigation will be certain families of schemes and stacks over S that are endowed with an action of the tangent sheaf T_S of S . We recall some standard facts and constructions that formalize the notion of a T_S action. Among the useful references are [Bernstein83], [Borel87] for \mathcal{D} -modules and [Beilinson-Drinfeld95], [Grothendieck68] [Berthelot74] [Illusie71] [Simpson95] and [Simpson97a] for \mathcal{D} -schemes and crystals.

2.1 \mathcal{D} -modules and local systems

Definition 2.1 *Let $F \rightarrow S$ be a vector bundle. An algebraic connection on F is a \mathbb{C} -morphism of sheaves $d^\nabla : F \rightarrow F \otimes \Omega_S^1$ satisfying $d^\nabla(f \cdot a) = f d^\nabla(a) + a \otimes df$ for every $f \in \mathcal{O}_S$ and $a \in F$.*

The differential d^∇ extends to $d^\nabla : F \otimes \Omega_S^i \rightarrow F \otimes \Omega_S^{i+1}$ by the Leibnitz rule:

$$d^\nabla(a \otimes \alpha) = a \otimes d\alpha + (-1)^i d^\nabla a \wedge \alpha.$$

Equivalently, a connection on F is a consistent way of lifting infinitesimal symmetries of S to infinitesimal symmetries of F that are linear along the fibers. The sheaf of infinitesimal symmetries of S is the holomorphic tangent bundle T_S . The sheaf $\mathcal{E}(F) \rightarrow S$ of the infinitesimal symmetries of F that are linear along the fibers can be described as follows. Put $\mathcal{D}_S^1(F)$ for the sheaf of differential operators on F of order ≤ 1 . There is a standard short exact sequence of \mathcal{O}_S -modules

$$0 \longrightarrow \text{End}(F) \longrightarrow \mathcal{D}_S^1(F) \xrightarrow{\sigma} \text{End}(F) \otimes T_S \longrightarrow 0,$$

where $\sigma(\partial) = [\partial, \bullet]$ is the principal symbol map. Then

$$\mathcal{E}(F) := \{ \partial \in \mathcal{D}_S^1(F) \mid \sigma(\partial) = \text{id}_F \otimes v, v \in T_S \}.$$

The sheaf $\mathcal{E}(F)$ has a natural \mathbb{C} -linear Lie bracket $[\partial', \partial''] = \partial' \partial'' - \partial'' \partial'$ and is called the *Atiyah algebra* of F . A connection on F is just a \mathcal{O}_S -linear splitting ∇ of the symbol sequence for $\mathcal{E}(F)$:

$$0 \longrightarrow \text{End}(F) \longrightarrow \mathcal{E}(F) \xrightarrow{\sigma} T_S \longrightarrow 0.$$

$\nwarrow \nabla$

The existence of an algebraic connection on F is obstructed by the extension class $e(F) \in H^1(S, \text{End}(F) \otimes \Omega_S^1)$ of this sequence, called the *Atiyah class* of F .

Definition 2.2 *A connection ∇ is called integrable if it is a morphism of sheaves of Lie algebras, i.e. if for any $\xi, \eta \in T_S$ we have $\nabla_{[\xi, \eta]} = [\nabla_\xi, \nabla_\eta]$. A vector bundle equipped with an integrable connection is called a local system.*

Equivalently, ∇ is integrable if $d^\nabla \circ d^\nabla = 0$. The algebraic curvature of ∇ is the endomorphism valued 2-form

$$\text{curv}(\nabla) = d^\nabla \circ d^\nabla \in H^0(S, \text{End}(F) \otimes \Omega_S^2).$$

It can be interpreted geometrically as the obstruction to lifting the T_S action on F to an action of the full sheaf \mathcal{D}_S of differential operators. In other words, a connection ∇ is integrable if and only if it lifts to an \mathcal{O}_S -linear morphism $\mathcal{D}_S \rightarrow \mathcal{D}_S(F)$. This implies in particular that the coherent sheaf of holomorphic sections of F is endowed with a left \mathcal{D}_S -action. Sometimes it is useful to consider more general objects of this type:

Definition 2.3 *A quasi-coherent sheaf on S equipped with a left \mathcal{D}_S -action is called a left \mathcal{D}_S -module.*

Denote by $\mathcal{M}_D^l(S)$ the category of all left \mathcal{D}_S -modules.

Remark 2.4 (i) One can consider *coherent* \mathcal{D}_S -modules. By definition these are locally finitely generated \mathcal{D}_S -modules. By an analogue of Oka's theorem [Bernstein83], [Borel87] the coherent \mathcal{D}_S -modules are always locally finitely presented.

There is also a notion of *smooth* \mathcal{D}_S -modules. By definition these are \mathcal{D}_S -modules that are coherent as \mathcal{O}_S -modules. It is a simple consequence [Bernstein83], [Borel87] of Kashiwara's lemma that the smooth \mathcal{D}_S -modules are precisely the ones that are locally free and of finite rank as \mathcal{O}_X -modules, i.e. the ones that arise from vector bundles with integrable connections.

(ii) There is a natural forgetful functor

$$\mathcal{M}_D^l(S) \xrightarrow{o} \mathcal{M}_O(S)$$

to the category $\mathcal{M}(S)$ of quasi-coherent \mathcal{O}_S -modules. This functor has a natural left adjoint functor

$$\begin{array}{ccc} \mathcal{M}_O(S) & \xrightarrow{\mathcal{D}_S \otimes} & \mathcal{M}_D^l(S) \\ F & \longrightarrow & \mathcal{D}_S \otimes_{\mathcal{O}_S} F \end{array}$$

(iii) There is another pair of adjoint functors - the first is the functor of horizontal sections

$$\begin{array}{ccc} \mathcal{M}_D^l(S) & \xrightarrow{\Gamma^{\text{hor}}} & \text{Vect}_{\mathbb{C}} \\ M & \longrightarrow & \text{Hom}_{\mathcal{M}_D^l(S)}(\mathcal{O}_S, M) \end{array},$$

where $\text{Vect}_{\mathbb{C}}$ is the category of vector spaces over \mathbb{C} . Its left adjoint is the functor assigning to a vector space the corresponding constant \mathcal{D} -module, i.e. the functor

$$\text{Vect}_{\mathbb{C}} \xrightarrow{\otimes_{\mathcal{O}_S}} \mathcal{M}_D^l(S).$$

For a smooth \mathcal{D} -module corresponding to a local system (M, ∇) the vector space of horizontal sections can be identified naturally with the space of all sections in M that are annihilated by d^∇ , i.e. $\Gamma^{\text{hor}}((M, \nabla)) = \{m \in \Gamma(S, M) \mid d^\nabla(m) = 0\}$.

(iv) $\mathcal{M}_D^l(S)$ is an abelian tensor category with a tensor product given by

$$M_1 \otimes M_2 := M_1 \otimes_{\mathcal{O}_S} M_2.$$

Clearly \mathcal{O}_S is a unit for \otimes and o is a tensor functor. The tensor product \otimes does not preserve coherency and the category $\mathcal{M}_D^l(S)$ does not have duals. Actually, it is easy to check that a \mathcal{D} -module M admits a dual iff M is smooth.

2.2 Crystals and local systems of schemes

The notions of a \mathcal{D} -module and a smooth \mathcal{D} -module readily generalize to families of schemes or stacks. In this section we gather some definitions and basic facts about those for further reference.

Definition 2.5 *Let $f : X \rightarrow S$ be a smooth morphism of smooth schemes. An algebraic connection on X/S is a consistent way of lifting of infinitesimal automorphisms of S to infinitesimal automorphisms of X , i.e. an \mathcal{O}_X -splitting ∇ of the exact sequence of vector bundles on S :*

$$(2.2.2) \quad 0 \longrightarrow T_f \longrightarrow T_X \xrightarrow{\quad df \quad} f^*T_S \longrightarrow 0.$$

$\nwarrow \nabla \nearrow$

where T_X and T_S are the tangent sheaves of X and S respectively and T_f is the sheaf of germs of vector fields on X that are tangent to the fibers of f .

The existence of an algebraic connection on X/S is obstructed by the extension class $e(X/S) \in H^1(X, T_f \otimes f^*\Omega_S^1)$ which we will call again the *Atiyah class* of X/S .

Remark 2.6 If f is proper and with connected fibers that do not have infinitesimal automorphisms, then the Atiyah class $e(X/S)$ is essentially the Kodaira-Spencer class of $f : X \rightarrow S$. Recall that the (naive) Kodaira-Spencer class $\kappa_{X/S}$ of a family $f : X \rightarrow S$ is the first edge homomorphism of the direct image sequence of (2.2.2), i.e.

$$0 \longrightarrow f_*T_f \longrightarrow f_*T_X \xrightarrow{\quad df \quad} f_*f^*T_S \xrightarrow{\quad \kappa_{X/S} \quad} R^1f_*T_f.$$

Since by assumption $f_*\mathcal{O}_X = \mathcal{O}_S$ we have that $f_*f^*T_S = T_S$ and thus $\kappa_{X/S}$ can be viewed as an element in $H^0(S, R^1f_*T_f \otimes \Omega_S^1)$. On the other hand, the Lerray spectral sequence gives

$$0 \longrightarrow H^1(f_*T_f \otimes \Omega_S^1) \longrightarrow H^1(X, T_f \otimes f^*\Omega_S^1) \longrightarrow H^0(R^1f_*T_f \otimes \Omega_S^1),$$

and due to the fact that $f_*T_f = 0$ we get an inclusion

$$H^1(X, T_f \otimes f^*\Omega_S^1) \hookrightarrow H^0(S, R^1f_*T_f \otimes \Omega_S^1)$$

under which $e(X/S)$ goes to $\kappa_{X/S}$.

Denote by the subsheaf $f^{-1}\mathcal{O}_S \subset \mathcal{O}_X$ consisting of germs of functions that are constant along the fibers. Let $f^{-1}T_S$ be the sheaf theoretic inverse image of the sheaf T_S and let $T_X^\sim \subset T_X$ be the centralizer of T_f with respect to the Lie bracket on T_X . There is a natural exact sequence of sheaves on X :

$$(2.2.3) \quad 0 \longrightarrow T_f \longrightarrow T_X^\sim \xrightarrow{df} f^{-1}T_S \longrightarrow 0.$$

which is just the pull-back of the extension (2.2.2) via the inclusion $f^{-1}T_S \hookrightarrow f^*T_S$. Moreover, since $T_f \triangleleft T_X^\sim$ is an ideal with respect to the \mathbb{C} -linear Lie bracket we get a natural bracket on $f^{-1}T_S$ and the exact sequence (2.2.3) becomes a sequence of Lie algebra sheaves. Under the assumptions of Remark 2.6 it is clear that the first edge homomorphism in the direct image of (2.2.3) is precisely the Kodaira-Spencer class of $f : X \rightarrow S$. Moreover if ∇ is a connection on X/S , then its restriction $\nabla|_{f^{-1}T_S} : f^{-1}T_S \rightarrow T_X^\sim$ is $f^{-1}\mathcal{O}_S$ -linear. Conversely, any $f^{-1}\mathcal{O}_S$ -linear splitting of the sequence (2.2.3) determines a connection on X/S when extended by \mathcal{O}_X -linearity.

Definition 2.7 *A connection ∇ on X/S is called integrable if it is a morphism of Lie algebra sheaves when restricted on $f^{-1}T_S$, i.e. if for any $\xi, \eta \in f^{-1}T_S$ we have $\nabla_{[\xi, \eta]} = [\nabla_\xi, \nabla_\eta]$. A smooth scheme X/S equipped with an integrable connection is called a local system of schemes.*

As in the vector bundle case the integrability of a connection on X/S ensures that the infinitesimal $f^{-1}T_S$ action on X given by ∇ lifts to a morphism of rings of differential operators $f^{-1}\mathcal{D}_S \rightarrow \mathcal{D}_X^\sim$ where \mathcal{D}_X^\sim is the centralizer of the sheaf of vertical differential operators \mathcal{D}_f in \mathcal{D}_X . It is again important to consider more general objects with such an action - the \mathcal{D}_S -schemes.

Definition 2.8 *A \mathcal{D}_S -algebra is an associative commutative unital algebra A in the category $\mathcal{M}_D^l(S)$. In other words it is a \mathcal{D}_S -module equipped with a horizontal associative commutative product and a horizontal section which is a unit for this product. The affine X -scheme $\text{Spec}(A)$ is called an affine \mathcal{D}_S -scheme.*

Denote by $\text{Comm}_{\mathcal{O}}(S)$ the category of unital associative commutative \mathcal{O} -algebras (= category of affine \mathcal{O} -schemes) and by $\text{Comm}_{\mathcal{D}}(S)$ the category of associative commutative \mathcal{D}_S -algebras (= category of affine \mathcal{D}_S -schemes).

Remark 2.9 (i) Both $\text{Comm}_{\mathcal{O}}(S)$ and $\text{Comm}_{\mathcal{D}}(S)$ are tensor categories w.r.t. the tensor product $A \otimes B := A \otimes_{\mathcal{O}_S} B$. Moreover \otimes is a coproduct and the natural forgetful functor

$$\text{Comm}_{\mathcal{O}}(S) \xrightarrow{o} \text{Comm}_{\mathcal{D}}(S)$$

commutes with \otimes .

(ii) A large supply of \mathcal{D}_S -algebras is provided by the jet construction. To any $R \in \text{Comm}_{\mathcal{O}}(S)$ one can associate a canonical \mathcal{D}_S algebra JR called the *jet algebra* of R [EGA4, 4_{IV} 16] by setting

$$JR := \text{Sym}(\mathcal{D}_S \otimes_{\mathcal{O}_S} R) \left/ \begin{array}{l} \text{the } \mathcal{D}_S\text{-ideal generated by } 1_R - 1 \\ \text{and } 1 \otimes r_1 r_2 - (1 \otimes r_1)(1 \otimes r_2) \text{ for} \\ \text{all } r_1, r_2 \in R \end{array} \right.$$

The functor

$$\text{Comm}_{\mathcal{O}}(S) \xrightarrow{J} \text{Comm}_{\mathcal{D}}(S)$$

is called the *jet functor* and also commutes with the tensor structure. It is left adjoint to the forgetful functor o . Geometrically JR can be interpreted as follows. Denote by $S^{(n)}$ the n -th infinitesimal neighborhood of the diagonal $S \xrightarrow{\Delta} S \times S$. In other words $S^{(n)}$ is a nilpotent scheme supported on S with $\mathcal{O}_{S^{(n)}} := \mathcal{O}_{S \times S} / I^{n+1}$ where I is the ideal of $\Delta(S)$ in $S \times S$. The structure sheaf $\mathcal{O}_{S^{(n)}}$ carries two \mathcal{O}_S -structures coming from the two embeddings of \mathcal{O}_S in $\mathcal{O}_{S \times S}$. Let B be an \mathcal{O}_S -algebra, then B -points of the formal scheme $\text{Spec}(oJB)$ are given by [EGA4, 4_{IV} 16], [Beilinson-Drinfeld95]:

$$\text{Hom}_{\mathcal{O}_S}(oJR, B) = \varprojlim \text{Hom}_{\mathcal{O}_{S^{(n)}}}(\mathcal{O}_{S^{(n)}} \otimes_{\mathcal{O}_S} R, B \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{(n)}}).$$

In particular we have

$$\mathbb{C} - \text{points of } \text{Spec}(oJR) = \left\{ (s, \gamma) \left| \begin{array}{l} s \in S, \gamma \text{ is a section of } \text{Spec}(R/S) \\ \text{over the complete formal neighborhood of } s \text{ in } S. \end{array} \right. \right\}$$

or equivalently $\text{Spec}(oJR)$ is the space of infinite jets of sections of $\text{Spec}(R)$ over S .

This notions can be globalized (see e.g. [Beilinson-Drinfeld95], [Simpson95]). Denote by $(S \times S)^\wedge$ and $(S \times S \times S)^\wedge$ the formal neighborhood of the (small) diagonals in the second and third Cartesian product of S .

Definition 2.10 *A \mathcal{D}_S -scheme or a crystal of schemes on S is a scheme $F \rightarrow S$ together with an isomorphism*

$$\varphi : (F \times S)_{|(S \times S)^\wedge} \xrightarrow{\sim} (S \times F)_{|(S \times S)^\wedge}$$

satisfying the cocycle condition

$$p_{23}^*(\varphi)p_{12}^*(\varphi) = p_{13}^*(\varphi)$$

for the resulting isomorphisms between the restrictions of $F \times S \times S$ and $S \times S \times F$ on $(S \times S \times S)^\wedge$.

This notion generalizes in an obvious way to the relative situation [Simpson95], and we can talk about crystals on X/S . If in addition $F \rightarrow S$ is a vector bundle and the identification φ is a morphism of vector bundles we will call F a *crystal of vector bundles* on S . By reducing to the case when X is affine over S it is easy to check that a crystal of vector bundles on X/S is the same as a vector bundle on X with a relative integrable connection over S [Simpson95, Lemma 8.1].

Remark 2.11 Strictly speaking, according to the standard terminology [Grothendieck68, Appendix] the object F from Definition 2.10 should be called a *stratification of schemes* rather than a crystal of schemes. However, due to the infinitesimal lifting property, if S is smooth, or in the relative case, if X is smooth over S , the notions of a crystal and stratification coincide [Simpson95, Section 8]. Since we will be working only with smooth projective families we will suppress the distinction between a stratification and a crystal.

Remark 2.12 For technical reasons it is sometimes more convenient to restate the conditions in Definition 2.10 in terms of the action of a formal groupoid. This turns out to be extremely useful for understanding the Griffiths transversality condition for variations of non-abelian Hodge structures (see [Simpson97a, Section 8] and [Simpson99, Section 9]. Recall [Berthelot74] [Illusie71, Chapter 8] [Simpson97a, Section 7] that a *formal groupoid of smooth type* is a groupoid

$$(e : X \rightarrow N, N \xrightleftharpoons[t]{s} X, m : N \times_X N \rightarrow N)$$

in the category of formal schemes such that X is a scheme, $e(X)$ is the topological space underlying the formal scheme N , N is locally the completion of a scheme of finite type and s and t are formally smooth.

To any smooth scheme S of finite type one associates the formal groupoid $S_{\text{DR}} := (S \times S)^\wedge \rightrightarrows S$. By abuse of notation we will write S_{DR} for the corresponding quotient stack as well. Now it is clear that $F \rightarrow S$ is a local system of schemes if and only if F is actually a sheaf on S_{DR} or equivalently if and only if F is equipped with an action of the formal groupoid $(S \times S)^\wedge \rightrightarrows S$.

We conclude this section with a discussion of the analogies between the linear and non-linear \mathcal{D}_S -objects. Denote by (Sch/S) and $\text{Sch}_{\mathcal{D}}(S)$ the categories of \mathcal{O}_S and \mathcal{D}_S -schemes respectively.

Remark 2.13 (i) Since forgetting of the \mathcal{D}_S action and passing to jets commute with Zariski and étale localization we get a pair of adjoint functors

$$\text{Sch}_{\mathcal{D}}(S) \xrightleftharpoons[J]{o} (\text{Sch}/S).$$

In particular for any scheme X/S we get a \mathcal{D}_S -scheme JX .

(ii) By analogy with the linear situation we can speak about crystals over S that have good regularity properties - i.e. \mathcal{D}_S -schemes that are smooth and of finite type as \mathcal{O}_S -schemes. It is almost immediate that these are precisely the local systems of schemes over S . It will be natural to call such a scheme a smooth \mathcal{D}_S -scheme but we will refrain from that for two reasons. The first is that we don't want to introduce redundant terminology and the second is that this term is reserved for another intrinsic notion of the smoothness of a crystal.

We will not be using this intrinsic notion in the sequel but we just comment on it in order to avoid confusion and to stress the difference between local systems of schemes and smooth crystals. To define the latter [Beilinson-Drinfeld95] one first looks at the affine case. Let A be a \mathcal{D}_S -algebra. To understand smoothness we need to understand the sheaf of Kähler differentials in the context of the category of A -modules i.e. the category of quasi-coherent sheaves over X that are endowed with two compatible actions of A (as an algebra) and \mathcal{D}_S . This category admits a better description.

Consider $A[\mathcal{D}_S]$ - the sheaf of algebras on X that is determined uniquely by the properties: (a) $A[\mathcal{D}_S]$ is equipped with two algebraic embeddings $A \hookrightarrow A[\mathcal{D}_S]$ and $T_S \hookrightarrow A[\mathcal{D}_S]$; (b) as an algebra $A[\mathcal{D}_S]$ is generated by A and T_S with the only relations being the ones coming from the T_S action on A . By construction the inclusion $T_S \hookrightarrow A[\mathcal{D}_S]$ extends to an inclusion of algebras $\mathcal{D}_S \hookrightarrow A[\mathcal{D}_S]$. As an A - \mathcal{D}_S -bimodule $A[\mathcal{D}_S]$ is isomorphic to $A \otimes_{\mathcal{O}_S} \mathcal{D}_S$.

In the case of an affine \mathcal{D}_S -scheme which is a local system of schemes, we have already encountered the sheaf of algebras $A[\mathcal{D}_S]$. Namely $A[\mathcal{D}_S] = f_* \mathcal{D}_{\text{Spec}(A)}^\sim$, where $f : \text{Spec}(A) \rightarrow X$ is the structure morphism.

Notice that an A -module in the above sense is just a sheaf of $A[\mathcal{D}_S]$ -modules with respect to the natural algebra structure. Furthermore one checks that the finitely generated projective $A[\mathcal{D}_S]$ -modules localize properly [Beilinson-Drinfeld95]. Beilinson and Drinfeld define the notions of formal \mathcal{D}_S -smoothness and \mathcal{D}_S -smoothness for the affine \mathcal{D}_S -scheme $\text{Spec}(A)$ by requiring a suitable infinitesimal lifting property in the category of A -modules. Moreover, they show that $\text{Spec}(A)$ is \mathcal{D}_S -smooth if and only if the sheaf of Kähler differentials Ω_A is projective as a $A[\mathcal{D}_S]$ -module and $\text{Spec}(A)$ is smooth over S [Beilinson-Drinfeld95]. In particular the intrinsic notion of smoothness of a crystal X over S requires that X be infinite dimensional as a scheme over S and thus a local system of schemes of finite type is never smooth in this sense.

(iii) Let $\text{Comm}_{\mathbb{C}}$ be the category of all commutative algebras over \mathbb{C} (= category of affine schemes over \mathbb{C}). Again we have a pair of adjoint functors

$$\text{Comm}_{\mathbb{C}} \begin{array}{c} \xrightarrow{\otimes_{\mathcal{O}_X}} \\ \xleftarrow{\Gamma^{\text{hor}}} \end{array} \text{Comm}_{\mathcal{D}}(S).$$

Globally, if X/S is a crystal of schemes the horizontal sections of X over S are just the algebraic sections $a : S \rightarrow X$ for which the subscheme $a(S) \subset X$ is a sub-crystal. In the case when $(X/S, \nabla)$ is a local system of schemes a sub-local system is a variety $Y \subset X$, smooth over S and such that ∇ lifts infinitesimal symmetries of S to infinitesimal symmetries of X .

which at the points of Y preserve Y . In other words the composition

$$(f^*T_S)|_Y \xrightarrow{\nabla} (T_X)|_Y \longrightarrow N_{Y/X}$$

must be identically zero (here $N_{X/Y}$ denotes the normal bundle of Y in X). In particular an algebraic section $a : S \rightarrow X$ of f will be horizontal if and only if the following diagram

$$\begin{array}{ccc} a^*f^*T_S & \xrightarrow{a^*\nabla} & a^*T_X \\ \parallel & \nearrow da & \\ T_S & & \end{array}$$

commutes.

3 Nonabelian Hodge structures

The non-abelian analogues of the (p, p) -classes live in the first non-abelian de Rham cohomology spaces of a variety. In this section we review briefly Simpson's theory of non-abelian Hodge structures on such spaces.

3.1 Abelian and nonabelian Hodge theory

The (abelian) cohomology groups of a smooth projective X are endowed with extra linear-algebraic data - their Hodge structure. Heuristically the Hodge decomposition of $H^\bullet(X, \mathbb{C})$ can be thought of as a linearization of the geometry of X . Similarly, the Hodge decomposition on the first cohomology set of X with coefficients in a non-abelian reductive group linearizes X in a sense by replacing it with a family of abelian varieties.

In order to get a better perspective of the setup for our problem we list some analogous concepts and facts both in abelian and nonabelian settings.

A(i) For a complex variety X denote by $H_B^\bullet(X, \mathbb{C})$ and $H_{DR}^\bullet(X, \mathbb{C})$ Betti and de Rham cohomology rings of X with complex coefficients respectively. The space $H_B^1(X, \mathbb{C})$ can also be interpreted as the space of representations of the fundamental group of X into the additive group \mathbb{C} . That is

$$H_B^1(X, \mathbb{C}) = \text{Hom}(\pi_1(X), \mathbb{C}) = \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}) = H^1(\pi_1(X), \mathbb{C}).$$

The de Rham theorem identifies the vector spaces $H_B^\bullet(X, \mathbb{C})$ and $H_{DR}^\bullet(X, \mathbb{C})$. Algebraically it is best to think of the de Rham cohomology of X as the hypercohomology of the (holomorphic) de Rham complex

$$\Omega_X^\bullet := \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{\dim X},$$

i.e. $H_{DR}^\bullet(X, \mathbb{C}) = \mathbb{H}^\bullet(X, \Omega_X^\bullet)$. The de Rham theorem can be interpreted in these terms as follows. The holomorphic Poincare lemma implies that \mathbb{C} (thought as a complex concentrated

in degree zero) and Ω_X^\bullet are quasi-isomorphic and thus have isomorphic hypercohomology. In particular we get $H_B^\bullet(X, \mathbb{C}) = \mathbb{H}^\bullet(X, \Omega_X^\bullet)$.

Similarly we can define the Betti and de Rham cohomology of X with coefficients in \mathbb{C}^\times . Thinking of \mathbb{C} as the Lie algebra of the reductive group \mathbb{C}^\times and using the exponential map $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ we can “exponentiate” the de Rham complex

$$\begin{array}{ccccccc} \mathbb{C} & \longrightarrow & \mathcal{O}_X & \xrightarrow{d} & \Omega_X^1 & \xrightarrow{d} & \cdots \xrightarrow{d} \Omega_X^{\dim X} \\ \exp \downarrow & & \exp \downarrow & & \parallel & & \parallel \\ \mathbb{C}^\times & \longrightarrow & \mathcal{O}_X^\times & \xrightarrow{d \log} & \Omega_X^1 & \xrightarrow{d} & \cdots \xrightarrow{d} \Omega_X^{\dim X} \end{array}$$

and so interpret the de Rham cohomology with coefficients in \mathbb{C}^\times as the hypercohomology of the logarithmic de Rham complex² (here \mathcal{O}_X is placed in degree zero)

$$\mathbb{C}(\dim X + 1)[1] := \mathcal{O}_X^\times \xrightarrow{d \log} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{\dim X}.$$

In particular $H_{\text{DR}}^\bullet(X, \mathbb{C}^\times) = \mathbb{H}^\bullet(X, \mathbb{C}(\dim X + 1)[1])$. A simple calculation with Čech cocycles now shows that in degree one we have

$$H_B^1(X, \mathbb{C}^\times) := \text{Hom}(\pi_1(X), \mathbb{C}^\times) = H^1(\pi_1(X), \mathbb{C}^\times),$$

and $H_{\text{DR}}^1(X, \mathbb{C}^\times)$ is the space of algebraic local systems of rank one on X . The analogue of the de Rham theorem in this case is the abelian Riemann-Hilbert correspondence which establishes an analytic isomorphism between $H_{\text{DR}}^1(X, \mathbb{C}^\times)$ and $H_B^1(X, \mathbb{C}^\times)$. Notice also that the exponential map $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ induces a surjective homomorphism from $H_{\text{DR}}^1(X, \mathbb{C})$ to the identity component $H_{\text{DR}}^1(X, \mathbb{C}^\times)_0$ of $H_{\text{DR}}^1(X, \mathbb{C}^\times)$.

NA(i) To any variety X one can associate the spaces $H_B^1(X, \text{GL}_n(\mathbb{C}))$ and $H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))$ - the first Betti and de Rham cohomology of X with coefficients in $\text{GL}_n(\mathbb{C})$. They are defined by analogy with **A(i)**. $H_B^1(X, \text{GL}_n(\mathbb{C})) := \text{Hom}(\pi_1(X), \text{GL}_n(\mathbb{C})) // \text{GL}_n(\mathbb{C})$ is the moduli space of semi-simplifications of representations of $\pi_1(X)$ in $\text{GL}_n(\mathbb{C})$ and $H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))$ is defined as the moduli space of rank n algebraic local systems on X . The general Riemann-Hilbert correspondence [Deligne70] [Simpson95, Proposition 7.8] gives an isomorphism

$$\psi_X : H_B^1(X, \text{GL}_n(\mathbb{C})) \longrightarrow H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))$$

of complex analytic spaces.

A(ii) One can also consider the Dolbeault cohomology

$$H_{\text{Dol}}^w(X, \mathbb{C}) := \oplus_{p+q=w} H^p(X, \Omega_X^q) = \mathbb{H}^w(X, \mathcal{O}_X \xrightarrow{0} \Omega_X^1 \xrightarrow{0} \cdots \xrightarrow{0} \Omega_X^{\dim X})$$

of X and its multiplicative version

$$H_{\text{Dol}}^\bullet(X, \mathbb{C}^\times) := \mathbb{H}^\bullet(X, \mathcal{O}_X^\times \xrightarrow{0} \Omega_X^1 \xrightarrow{0} \cdots \xrightarrow{0} \Omega_X^{\dim X}).$$

²The slightly odd notation used here is standard in the theory of Deligne cohomology.

Concretely in degree one we have

$$H_{\text{Dol}}^1(X, \mathbb{C}) = H^1(\mathcal{O}_X) \oplus H^0(\Omega_X^1)$$

and

$$H_{\text{Dol}}^1(X, \mathbb{C}^\times) = H^1(\mathcal{O}_X^\times) \times H^0(\Omega_X^1) = T^\vee H^1(\mathcal{O}_X^\times),$$

respectively.

NA(ii) To any polarized variety $(X, \mathcal{O}_X(1))$ one can associate the first Dolbeault cohomology set $H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))$ of X with coefficients in $\text{GL}_n(\mathbb{C})$. It is defined by analogy with **A(ii)** as an appropriate moduli space.

Definition 3.1 *A Higgs bundle on X is a pair (E, θ) consisting of a vector bundle E of rank n and a homomorphism $\theta : E \rightarrow E \otimes \Omega_X^1$ satisfying the symmetry condition $\theta \wedge \theta = 0$. A Higgs bundle (E, θ) is called $\mathcal{O}_X(1)$ -stable (semistable) if for every θ invariant subsheaf $F \subset E$ one has $p(F) < (\leq) p(E)$. Here $p(F) := \chi(X, F \otimes \mathcal{O}_X(n)) / \text{rk}(F)$ is the reduced Hilbert polynomial of F .*

The first Dolbeault cohomology of X with coefficients in $\text{GL}_n(\mathbb{C})$ is by definition the moduli space of semistable rank n Higgs bundles on X with vanishing c_1 and c_2 . It has a component birationally equivalent to the cotangent bundle $T^\vee H^1(X, \text{GL}_n(\mathcal{O}_X))^{\text{reg}}$ to the regular locus of the moduli space $H^1(\text{GL}_n(\mathcal{O}_X))$ of semistable vector bundles of rank n and trivial c_1 and c_2 .

A(iii) There is an equivalence between the Dolbeault and the De Rham cohomology which depends only on the class of the chosen Kähler metric (polarization). More precisely we have the following

Theorem (Hodge theorem). *For any Kähler X and any $k : 0 \leq k \leq \dim_{\mathbb{R}} X$ there is a natural isomorphism*

$$\tau_X : H_{\text{DR}}^k(X, \mathbb{C}) \xrightarrow{\cong} H_{\text{Dol}}^k(X, \mathbb{C}) := \bigoplus_{p+q=k} H^q(X, \Omega_X^p).$$

The isomorphism τ_X is built in two steps. First one shows that both de Rham and Dolbeault cohomology classes are represented by harmonic forms and then one uses the Kähler identities to identify the harmonic representatives.

Furthermore, the exponential map $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ combined with the Hodge theorem gives an isomorphism between the multiplicative De Rham and Dolbeault cohomology. In particular there is an isomorphism

$$\tau_X : H_{\text{DR}}^1(X, \mathbb{C}^\times)_0 \xrightarrow{\cong} H_{\text{Dol}}^1(X, \mathbb{C}^\times)_0.$$

Explicitly one has the Cartan decomposition $\mathbb{C}^\times = S^1 \times \mathbb{R}^+$ which induces

$$\tau_X : H_{\text{DR}}^1(X, \mathbb{C}^\times)_0 \rightarrow H^1(X, S^1)_0 \times H^0(\Omega_X^1) \cong H^1(\mathcal{O}_X^\times)_0 \times H^0(\Omega_X^1).$$

NA(iii) There is an equivalence between the de Rham and Dolbeault moduli spaces.

Theorem ([Corlette88], [Simpson92]). *For any smooth projective variety X there is a natural homeomorphism*

$$\tau_X : H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C})) \xrightarrow{\cong} H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C})).$$

Similarly to A(iii) the isomorphism τ_X is built by means of harmonic representatives. The latter are the so called harmonic bundles, i.e. triples (F, ∇, h) where F is a holomorphic bundle of rank n , ∇ is a flat holomorphic connection on F and h is a hermitian metric on F s.t. the corresponding $\pi_1(X)$ -equivariant map $\tilde{h} : \tilde{X} \rightarrow \text{GL}_n(\mathbb{C})/U(n)$ has minimal energy.

For future reference the isomorphisms τ_X and ψ_X can be combined to yield an isomorphism

$$\phi_X : H_{\text{B}}^1(X, \text{GL}_n(\mathbb{C})) \longrightarrow H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C})).$$

A(iv) Due to the isomorphism $H_{\text{Dol}}^1(X, \mathbb{C}^\times) \cong T^\vee H^1(\mathcal{O}_X^\times)$ the first Dolbeault cohomology group can be viewed as an algebraic symplectic manifold. Moreover, the Hodge decomposition induces two transversal Lagrangian fibrations

$$\begin{array}{ccc} & H_{\text{Dol}}^1(X, \mathbb{C}^\times) & \\ \swarrow & & \searrow \\ H^1(\mathcal{O}_X^\times) & & H^0(\Omega_X^1) \end{array}$$

parameterized by the $(0, 1)$ and $(1, 0)$ parts of the Hodge structure respectively.

NA(iv) The first non-abelian Dolbeault cohomology group is an algebraic symplectic manifold [Biswas94], [Donagi-Markman96]. This symplectic structure restricts to the standard one on $T^\vee H^1(\text{GL}_n(\mathcal{O}_X))$ and the projection on $H^1(\text{GL}_n(\mathcal{O}_X))$ and the Hitchin map give two generically transversal Lagrangian fibrations [Arapura97].

$$\begin{array}{ccc} & H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C})) & \\ \swarrow & & \searrow^h \\ H^1(\text{GL}_n(\mathcal{O}_X)) & & B_X := \oplus_{i=1}^n H^0(S^i \Omega_X^1) \end{array}$$

By analogy with **A(iv)** the spaces $H^1(\mathrm{GL}_n(\mathcal{O}_X))$ and B_X should be interpreted as the $(0, 1)$ and the $(1, 0)$ part of the non-abelian Hodge structure.

A(v) The locus of Hodge classes in $H_{\mathrm{DR}}^{2p}(X, \mathbb{C})$ is the locus of all integral cohomology classes whose image under τ_X is of type (p, p) . In other words the locus of Hodge classes is just the set $\mathcal{V}_{\mathbb{Z}}^p := \tau_X^{-1}(\phi_X(H_{\mathrm{B}}^{2p}(X, \mathbb{Z})) \cap H^p(X, \Omega_X^p))$. The Hodge conjecture asserts that up to tensoring by \mathbb{Q} the Hodge classes are exactly the classes of algebraic cycles and hence the importance of this locus. The first crude approximation to the $\mathcal{V}_{\mathbb{Z}}^p$ is obtained by dropping the integrality restriction. In this way we arrive at the locus of all (p, p) classes: $\mathcal{V}_{\mathbb{C}}^p := \tau_X^{-1}(H^p(X, \Omega_X^p))$.

Sometimes it is convenient to describe the (formal) loci of Hodge type as invariants of certain group of symmetries of the Hodge structure. The *Hodge group* $\mathbf{Hod}_{2p}(X)$ is a subgroup of $\mathrm{GL}(H_{\mathrm{B}}^{2p}(X, \mathbb{Q}))$ such that

$$\mathcal{V}_{\mathbb{Z}}^p \otimes \mathbb{Q} = H_{\mathrm{B}}^{2p}(X, \mathbb{Q})^{\mathbf{Hod}_{2p}(X)}.$$

It can be defined as follows. The natural rescaling action of $U(1)$ on the cotangent bundle of X induces action on the (p, q) forms and through the Hodge decomposition and the de Rham theorem an action on $H_{\mathrm{B}}^{2p}(X, \mathbb{Q}) \otimes \mathbb{C} = H_{\mathrm{B}}^{2p}(X, \mathbb{C})$. This action can be extended to a homomorphism

$$C : \mathbb{C}^{\times} \longrightarrow \mathrm{GL}(H_{\mathrm{B}}^{2p}(X, \mathbb{C})).$$

Explicitly an element $t \in \mathbb{C}^{\times}$ acts on the piece $H^p(X, \Omega_X^q)$ as multiplication by $C_t = t^{q-p}$ (Weil operators).

The Hodge group $\mathbf{Hod}_{2p}(X)$ then is defined as the smallest subgroup in $\mathrm{GL}(H_{\mathrm{B}}^{2p}(X, \mathbb{C}))$ that is defined over \mathbb{Q} and contains $C(\mathbb{C}^{\times})$. It is not hard to see then that

$$\begin{aligned} \mathcal{V}_{\mathbb{C}}^p &= H_{\mathrm{B}}^{2p}(X, \mathbb{C})^{\mathbb{C}^{\times}} \\ \mathcal{V}_{\mathbb{Z}}^p \otimes \mathbb{Q} &= H_{\mathrm{B}}^{2p}(X, \mathbb{Q})^{\mathbf{Hod}_{2p}(X)} \end{aligned}$$

NA(v) Recall first the following

Definition 3.2 *A polarized integral variation of Hodge structures on X consists of the following data: a) local system (V, ∇) on X ; b) a lattice $V_{\mathbb{Z}} \subset V$; c) a finite decreasing filtration $F^{\bullet} : \dots \subseteq F^i \subseteq F^{i-1} \subseteq \dots \subseteq V$ such that \overline{F}^{\bullet} is the complex conjugate filtration to F^{\bullet} (w.r.t. the real structure coming from the inclusion of $V_{\mathbb{Z}}$ in V), then there exists a number w for which $V = F^i \oplus \overline{F}^{w-i+1}$ for all i ; d) a horizontal hermitian form ψ on V . Furthermore, these objects should satisfy the following axioms:*

(ZV1) *(Holomorphicity) For all i $F^i \subset V$ is a holomorphic subbundle.*

(ZV2) (*Griffiths transversality*) The filtration F^\bullet satisfies

$$d^\nabla : F^i \rightarrow F^{i-1} \otimes \Omega_X^1$$

for all i .

(ZV3) (*Polarization*) Let $V^{p,q} := F^p \cap \overline{F}^q$. Then the natural C^∞ decomposition $V = \bigoplus_{p+q=w} V^{p,q}$ is ψ -orthogonal and $\psi|_{V^{p,q}}$ is positive definite for p -even and negative definite for p -odd.

(ZV4) (*Integrality*) The lattice $V_{\mathbb{Z}}$ is horizontal.

Remark 3.3 In general, two exhaustive decreasing filtrations F^\bullet and \underline{F}^\bullet on a complex vector space V are called w -opposed if $V = F^i \oplus \underline{F}^{w-i+1}$ for all i .

The non-abelian analogue of the Hodge cohomology classes are the integral variations of Hodge structures. Thus the nonabelian counterpart of the group $\mathcal{V}_{\mathbb{Z}}^p$ is the locus

$$\mathcal{V}_{\mathbb{Z}} \subset H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C})),$$

consisting of all local systems underlying polarizable integral variations of Hodge structures.

It is very hard to describe this locus geometrically. The only information about the properties $\mathcal{V}_{\mathbb{Z}}$ in general is provided by an Arakelov type theorem proven originally by Faltings for variations of Hodge structures of weight one [Faltings83] and in the general case by Deligne [Deligne87]. According to this theorem $\mathcal{V}_{\mathbb{Z}}$ is a finite set which makes it even harder to characterize geometrically. The crudest geometric approximation of $\mathcal{V}_{\mathbb{Z}}$ in this case is the locus $\mathcal{V}_{\mathbb{C}} \subset H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))$ consisting of all local systems underlying polarizable complex variations of Hodge structures, i.e. variations satisfying all of the above properties with the exception of the integrality assumption. More precisely we have the following

Definition 3.4 ([Deligne87],[Simpson92]) A complex variation of Hodge structures of weight w on a smooth projective X is a complex local system \mathbb{V} on X together with a flat hermitian form ψ on \mathbb{V} so that the fibers \mathbb{V}_x , $x \in X$ are furnished with a decomposition $\mathbb{V}_x = \bigoplus_{p \in \mathbb{Z}} \mathbb{V}_x^p$ satisfying the following axioms

(CV1) (*Holomorphicity*) The subspaces $F^p = \bigoplus_{i \geq p} \mathbb{V}_x^i$ and $\overline{F}^q = \bigoplus_{i \leq w-q} \mathbb{V}_x^i$ vary with x holomorphically and anti-holomorphically respectively.

(CV2) (*Griffiths transversality*) If v is a local differentiable section of \mathbb{V} which is contained in F^p (\overline{F}^q), then the Lie derivative of v with respect to any vector field on X is contained in F^{p-1} (\overline{F}^{q-1})

(CV3) (*Polarization*) The decomposition $\mathbb{V}_x = \bigoplus_{p \in \mathbb{Z}} \mathbb{V}_x^p$ is ψ -orthogonal and $\psi|_{\mathbb{V}_x^p}$ is positive definite for p even and negative definite for p odd.

The analogue of the Hodge group for the first non-abelian cohomology of X would be a subgroup $\mathbf{Hod}_{\text{na}}(X) \subset \text{Aut}(H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C})))$ in the group of algebraic automorphisms of the Dolbeault moduli space with the property

$$\phi_X(\mathcal{V}_{\mathbb{Q}}) = H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))^{\mathbf{Hod}_{\text{na}}(X)},$$

where $\mathcal{V}_{\mathbb{Q}}$ is the locus of \mathbb{Q} -variations of Hodge structures. As a preliminary step one would like to have good analogues of the Weil operators for the non-abelian cohomology. They were found by C. Simpson:

Theorem ([Simpson92]). *Consider the standard \mathbb{C}^\times action on the Dolbeault moduli space given by*

$$\begin{aligned} C : \quad \mathbb{C}^\times &\longrightarrow \text{Aut}(H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))) \\ t &\longrightarrow ((E, \theta) \mapsto (E, t\theta)). \end{aligned}$$

Then the locus of complex variations of Hodge structures coincides with the fixed-point set of the action C , i.e.

$$\tau_X(\mathcal{V}_{\mathbb{C}}) = H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))^{\mathbb{C}^\times}.$$

Thus the nonabelian (p, p) classes are just local systems whose Higgs bundles are fixed under the \mathbb{C}^\times -action.

Remark 3.5 The group $\mathbf{Hod}_{\text{na}}(X)$ is more elusive. It is clear that $\mathbf{Hod}_{\text{na}}(X)$ should consist of automorphisms of $H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))$ that preserve the non-abelian Hodge decomposition - that is, automorphism that preserve the rational map to the stack of all semistable rank n vector bundles: $q : H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C})) \rightarrow H^1(X, \text{GL}_n(\mathcal{O}_X))$. More precisely, consider the Mumford-type group

$$\mathcal{G}_n(X) := \left\{ (g, s) \left| \begin{array}{l} \text{where } g \in \text{Aut}(H^1(X, \text{GL}_n(\mathcal{O}_X))) \text{ and } s : \\ H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C})) \rightarrow H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C})) \text{ is an} \\ \text{automorphism that is linear on the fibers of } q \\ \text{and induces } g \text{ on } H^1(X, \text{GL}_n(\mathcal{O}_X)) \end{array} \right. \right\}.$$

On the other hand we have a natural central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathcal{G}_n^{\text{Weil}}(X) \longrightarrow \text{Aut}(H^1(X, \text{GL}_n(\mathcal{O}_X))) \longrightarrow 1,$$

consisting of the automorphisms that preserve q and act by Weil operators on its fibers.

It is clear that $\mathcal{G}_n^{\text{Weil}}(X) \subset \mathcal{G}_n(X)$ and we can define $\mathbf{Hod}_{\text{na}}(X)$ to be the minimal subgroup of $\mathcal{G}_n(X)$ that is defined over the \mathbb{Q} and contains $\mathcal{G}_n^{\text{Weil}}(X)$. This definition is analogous to the one we have in the abelian case but there is no evidence that it is the right one.

3.2 Twistors and the Hodge filtration

From the view-point of the Hodge-de Rham spectral sequence the (abelian) cohomology groups of a projective manifold come naturally equipped with the Hodge filtration rather than the Hodge decomposition. This observation becomes very important when one studies variations of Hodge structures. To understand the analogue of the Hodge filtration on the first non-abelian cohomology one needs a geometric interpretation of the latter due to Deligne [Deligne89], Deninger [Deninger91] and Simpson [Simpson89], [Simpson91], [Simpson97a] which we proceed to describe.

A(vi) Suppose V is a finite dimensional complex vector space furnished with an exhaustive decreasing filtration F^\bullet . Simpson defines the *Rees module* corresponding to (V, F^\bullet) to be the quasi coherent sheaf $\xi(V, F)$ over \mathbb{A}^1 given by

$$\xi(V, F) = \sum_p \lambda^{-p} F^p V \otimes \mathcal{O}_{\mathbb{A}^1} \subset V \otimes \mathcal{O}_{\mathbb{C}^\times},$$

where λ is the coordinate on \mathbb{A}^1 . Equivalently $\xi(V, F)$ is the sheafification of the $\mathbb{C}[\lambda]$ -module $\bigoplus_p \lambda^{-p} F^p$. The sheaf $\xi(V, F)$ is locally free over \mathbb{A}^1 and is equipped with a natural \mathbb{C}^\times -action covering the action on \mathbb{A}^1 , namely the element $t \in \mathbb{C}^\times$ acts by t^p on the piece $\lambda^{-p} F^p V$. Furthermore $\xi(V, F)$ is provided with an identification between its fiber at $1 \in \mathbb{C}^\times$ and V . The fiber of $\xi(V, F)$ at $0 \in \mathbb{A}^1$ is the associated graded space of V , so one way to think of $\xi(V, F)$ is as a canonical deformation of V to $\text{Gr}_F(V)$.

Conversely if \mathcal{V} is a quasi-coherent \mathbb{C}^\times -sheaf on \mathbb{A}^1 , then $\mathcal{V}|_{\mathbb{C}^\times} \cong \mathcal{V}_1 \otimes \mathcal{O}_{\mathbb{C}^\times}$ and we obtain a filtration F^\bullet on \mathcal{V}_1 by putting $F^p = \{v \in \mathcal{V}_1 | v \otimes \lambda^{-p} \in \mathcal{V}\}$. If, in addition, \mathcal{V} is locally free, then the natural map $\xi(\mathcal{V}_1, F) \rightarrow \mathcal{V}$ is an isomorphism.

This construction is compatible with \oplus , \otimes , Hom and passing to duals. A map $f : W \rightarrow V$ of filtered vector spaces induces a morphism $\xi(f) : \xi(V, F^\bullet V) \rightarrow \xi(W, F^\bullet W)$ of locally free \mathbb{C}^\times -sheaves which respects kernels, i.e. $\ker(\xi(f)) = \xi(\ker(f))$. The map f is strictly compatible with the filtrations iff $\xi(f)$ is a morphism of vector bundles and in this case $\text{coker}(\xi(f)) = \xi(\text{coker}(f))$.

If \underline{F}^\bullet is another filtration on V , then we may apply the same construction at ∞ and glue the resulting sheaves over $\mathbb{C}^\times \subset \mathbb{A}^1$ to obtain a locally free \mathbb{C}^\times sheaf $\xi(V, F, \underline{F})$ with a fiber over 1 equal to V . Suppose F^\bullet and \underline{F}^\bullet are w -opposed filtrations (cf. remark 3.3), then $\xi(V, F, \underline{F})$ is semistable of slope w , i.e. is a direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(w)$. If, in addition, V has a real structure given by an anti-holomorphic involution ρ and if $\underline{F} = \rho(F)$, then $\xi(V, F, \underline{F})$ is provided with an antilinear involution σ covering the antipodal involution $\sigma_{\mathbb{P}^1}(\lambda) := -\bar{\lambda}^{-1}$ on \mathbb{P}^1 and restricting to ρ on the fiber at 1.

To summarize, the Rees module construction ξ provides the following equivalences of tensor categories [Simpson97b, Proposition 1.2 and Section 2] (see also [Kaledin97, Appendix]):

$$\left\{ \begin{array}{l} \text{The category with objects -} \\ \text{complex vector spaces with} \\ \text{exhaustive decreasing filtra-} \\ \text{tions and with morphisms} \\ \text{- filtration preserving linear} \\ \text{maps} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{The category of locally free} \\ \text{sheaves on } \mathbb{A}^1 \text{ equipped with a} \\ \mathbb{C}^\times \text{ action covering the standard} \\ \text{action on } \mathbb{A}^1 \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{The category with objects -} \\ \text{complex vector spaces with} \\ \text{exhaustive decreasing filtra-} \\ \text{tions and with morphisms -} \\ \text{linear maps strictly preserv-} \\ \text{ing the filtrations} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{The category of } \mathbb{C}^\times\text{-equivariant} \\ \text{vector bundles on } \mathbb{A}^1 \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{The category of complex} \\ \text{vector spaces equipped with} \\ \text{two decreasing filtrations } F \\ \text{and } \bar{F} \text{ with morphisms - lin-} \\ \text{ear maps preserving the two} \\ \text{filtrations} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{The category of } \mathbb{C}^\times\text{-equivariant} \\ \text{locally free sheaves on } \mathbb{P}^1. \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{The category of pure com-} \\ \text{plex Hodge structures} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{The category of } \mathbb{C}^\times\text{-equivariant} \\ \text{vector bundles on } \mathbb{P}^1 \end{array} \right\}$$

Moreover in this case pure Hodge structures of weight w correspond to semistable bundles of slope w . Finally, we have the equivalence

$$\left\{ \begin{array}{l} \text{The category of } \mathbb{R}\text{-Hodge} \\ \text{structures} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{The category of } \mathbb{C}^\times\text{-equivariant} \\ \text{vector bundles on } \mathbb{P}^1 \text{ equipped} \\ \text{with an anti-linear involution} \\ \text{covering the antipodal involution} \\ \text{on } \mathbb{P}^1 \end{array} \right\}$$

The next crucial observation of Deligne [Deligne89] is that if $(V_{\mathbb{R}} \subset V, F^\bullet)$ is an \mathbb{R} -Hodge structure of weight one, then the space $V \cong F^1 V \otimes_{\mathbb{R}} \mathbb{C}$ has a canonical action of the quaternions \mathbb{H} given by

$$\begin{aligned} I(f \otimes a) &= f \otimes \sqrt{-1}a \\ J(f \otimes a) &= (-\sqrt{-1}f) \otimes \bar{a} \\ K(f \otimes a) &= IJ(f \otimes a) = -(\sqrt{-1}f) \otimes (\sqrt{-1}\bar{a}) \end{aligned}$$

for a $f \otimes a \in F^1 V \otimes_{\mathbb{R}} \mathbb{C}$.

But for a quaternionic vector space V (or more generally for a pseudo-quaternionic³ manifold M) one has the so called twistor construction which puts together all the complex structures on V (respectively M) coming from the \mathbb{H} -action. This construction goes as follows. Let \mathbb{S} be the two sphere of pure quaternions of norm 1 with the complex structure at a point $q \in \mathbb{S}$ given by the left multiplication with q . In other words after identifying the tangent space $T_{\mathbb{S},q}$ with all pure quaternions $\perp q$ the multiplication by $i \in \mathbb{C}$ is given by $q \cdot$. Let M be a pseudo quaternionic manifold. Its twistor space is $Z = M \times \mathbb{S}$ with the complex structure on the tangent space $T_{M \times \mathbb{S},(m,q)} = T_m \times T_q$ given by $(q \cdot, i \cdot)$. If all the almost complex structures on M induced by the \mathbb{H} -action are integrable (such manifolds are called *hypercomplex* [Boyer88] [Kaledin96]), then Z is a complex manifold and one has: a) $\zeta : Z = M \times \mathbb{S} \rightarrow \mathbb{S}$ is holomorphic; b) for any $m \in M$ the assignment $q \rightarrow (m, q)$ is a holomorphic section; c) the map $(m, q) \rightarrow (m, -q)$ is an antiholomorphic involution.

Identify \mathbb{S} with \mathbb{P}^1 so that $(J, I, -J)$ are identified with $(0, 1, \infty)$. From all we said above it is now clear that for a real Hodge structure of weight one the canonical deformation $\xi(V, F)$ of the Hodge filtration to the Hodge decomposition is precisely the twistor space of the quaternionic vector space V .

Remark 3.6 If a pseudo-quaternionic manifold M possesses a Riemannian metric g such that the corresponding Hermitian two form $g(\bullet, q\bullet)$ is closed for all almost complex structures q , then all the q 's are integrable and g is a Kähler metric in any of them. Such manifolds are called [Calabi79], [Besse87, Chapter 14] *hyperkähler* and can be characterized [Hitchin *et al.*87] from the twistor viewpoint as follows. Let $\Omega_{\zeta}^2 \rightarrow Z$ be the sheaf of holomorphic two forms along the fibers of ζ . Then a pseudo quaternionic manifold M is hyperkähler iff its twistor space Z is a complex manifold and there exists a section $\Omega \in H^0(Z, \Omega_{\zeta}^2 \otimes \zeta^* \mathcal{O}_{\mathbb{P}^1}(2))$ which becomes a holomorphic symplectic form when restricted on any fiber of ζ .

To fit this into the above picture notice that a choice of a polarization ψ for a \mathbb{R} -Hodge structure of weight one corresponds to a hyperkähler structure on V with the metric being the flat metric given by ψ .

NA(vi) The construction ξ of Simpson and its relation to the pseudo-quaternionic and the hyperkähler picture give a way of understanding the analogue of the Hodge filtration for the first nonabelian cohomology [Deligne89], [Simpson97a], [Simpson91].

The starting point is the following beautiful observation of Hitchin. Denote by $H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))^{\text{reg}}$ and $H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))^{\text{reg}}$ the smooth loci of the de Rham and Dolbeault cohomology spaces. Then the homeomorphism τ_X given by the nonabelian Hodge theorem (see NA(iii)) restricts to a smooth isomorphism

$$H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))^{\text{reg}} = H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))^{\text{reg}} =: M^{\text{reg}}.$$

³Recall that a manifold M is called pseudo-quaternionic [Besse87, Chapter 14] if its tangent bundle is equipped with a linear action of the quaternions $\mathbb{H} \times T_M \rightarrow T_M$

Furthermore if I and J denote the complex structures coming from

$H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))^{\text{reg}}$ and $H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))^{\text{reg}}$ respectively, then $K = IJ$ is also a complex structure and this triple gives rise to a hypercomplex structure on M^{reg} . Finally the natural polarization on $H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))^{\text{reg}}$ coming from the categorical quotient construction is a hyperkähler metric. This was proven by Hitchin [Hitchin87] in the case when X is a curve and by Deligne [Deligne89] and Fujiki [Fujiki91] in general.

In view of this and the discussion in A(vi) it is reasonable to try to interpret the Hodge filtration on $H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))$ as a special twistor deformation. The main problem is how to deal with the singularities of the moduli space.

The sheaf \mathcal{D}_X of differential operators on X is naturally filtered: F^{-p} = operators of order $\leq p$. We can build $\xi(\mathcal{D}_X, F)$ -a quasi coherent sheaf of algebras on $X \times \mathbb{A}^1$. For a $\lambda \in \mathbb{A}^1$ denote by i_λ the inclusion $i_\lambda : X \rightarrow X \times \{\lambda\} \subset X \times \mathbb{A}^1$. Define a λ -connection on X to be a sheaf E of left modules for the sheaf of algebras $i_\lambda^* \xi(\mathcal{D}_X, F)$ that is locally free as a $\mathcal{O}_{X \times \mathbb{A}^1}$ -module. Say that E is *semistable* if the Chern classes of E vanish and the degree of any subbundle is less than or equal to zero.

It is easy to see that a λ -connection is a vector bundle E on X together with a splitting ∇_λ of the twisted symbol sequence

$$0 \longrightarrow \text{End}(E) \longrightarrow \mathcal{E}(E) \begin{array}{c} \xrightarrow{\lambda\sigma} \\ \xleftarrow{\nabla_\lambda} \end{array} T_X \longrightarrow 0,$$

that is a morphism of sheaves of Lie algebras over \mathbb{C} . Equivalently a λ -connection is an operator $d^{\nabla_\lambda} : E \rightarrow E \otimes \Omega_X^1$ satisfying $d^{\nabla_\lambda}(ae) = e \otimes \lambda da + ad^{\nabla_\lambda}(e)$ and the integrability condition $d^{\nabla_\lambda} \circ d^{\nabla_\lambda} = 0$.

Notice that for $\lambda = 1$ a λ -connection is just a usual integrable connection. For $\lambda = 0$ a λ -connection is a Higgs bundle. Therefore this definition provides a deformation from the notion of connection to the notion of a Higgs bundle. If $\lambda \neq 0$, and ∇_λ is a ∇ -connection, then $\lambda^{-1}\nabla_\lambda$ is a usual integrable connection. In particular the stability of a λ -connection is automatic for $\lambda \neq 0$ and specializes to the stability for Higgs bundles for $\lambda = 0$.

The importance of the λ -connections comes from the resulting moduli spaces. Simpson had proved [Simpson94, Theorem 4.7], [Simpson97a, Proposition 4.1] that for any complex smooth projective variety X there exists a quasi projective coarse moduli space $H_{\text{Hod}}^1(X, \text{GL}(n, \mathbb{C})) \rightarrow \mathbb{A}^1$ of semi-stable λ -connections of rank n . The fibers

$$H_{\text{Hod}}^1(X, \text{GL}(n, \mathbb{C}))_1 \text{ and } H_{\text{Hod}}^1(X, \text{GL}(n, \mathbb{C}))_0$$

over $1, 0 \in \mathbb{A}^1$ respectively are the moduli spaces $H_{\text{DR}}^1(X, \text{GL}_n(\mathbb{C}))$ and $H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))$ and the natural action of \mathbb{C}^\times defined by $t : (E, \nabla_\lambda) \rightarrow (E, \nabla_{t\lambda} := t\nabla_\lambda)$ is an algebraic action on $H_{\text{Hod}}^1(X, \text{GL}(n, \mathbb{C}))$ covering the standard action on \mathbb{A}^1 . Furthermore $H_{\text{Hod}}^1(X, \text{GL}(n, \mathbb{C}))$ can be glued to its complex conjugate [Simpson97a, Section 4], i.e. there is a complex analytic space $H_{\text{Del}}^1(X, \text{GL}(n, \mathbb{C}))$ over \mathbb{P}^1 characterized uniquely by the following properties:

- (a) There is a \mathbb{C}^\times action on $H_{\text{Del}}^1(X, \text{GL}(n, \mathbb{C}))$ covering the standard action on \mathbb{P}^1 and an anti-linear involution σ compatible with the \mathbb{C}^\times action and covering the antipodal involution $\lambda \rightarrow -\bar{\lambda}^{-1}$ on \mathbb{P}^1 ;

(b) There is an algebraic \mathbb{C}^\times -equivariant identification

$$H_{\text{Del}}^1(X, \text{GL}(n, \mathbb{C}))|_{\mathbb{A}^1} \cong H_{\text{Hod}}^1(X, \text{GL}(n, \mathbb{C}));$$

(c) On the fiber $H_{\text{Hod}}^1(\text{GL}_n(\mathbb{C}), 1) \cong H_{\text{B}}^1(X, \text{GL}_n(\mathbb{C}))$ the involution σ takes a representation to the dual of the complex conjugate representation.

A harmonic bundle yields a family of holomorphic bundles with λ -connections which extends to a holomorphic section (a twistor line) $\mathbb{P}^1 \rightarrow H_{\text{Hod}}^1(X, \text{GL}_n(\mathbb{C}))$. Let M be the moduli space of harmonic bundles. Deligne had shown [Deligne89], [Simpson97a, Section 4] that the trivialization $H_{\text{Del}}^1(X, \text{GL}_n(\mathbb{C})) \cong M \times \mathbb{P}^1$ given by the twistor lines is a homeomorphism and that on the set of smooth points it identifies $H_{\text{Del}}^1(X, \text{GL}_n(\mathbb{C}))^{\text{reg}}$ with the twistor space for the pseudo-quaternionic structure on M^{reg} .

In view of all this Simpson defines a *nonabelian filtration* to be a \mathbb{C}^\times -equivariant scheme over \mathbb{A}^1 . Thus the space $H_{\text{Hod}}^1(X, \text{GL}_n(\mathbb{C}))$ is interpreted as the nonabelian Hodge filtration and the space $\sigma(H_{\text{Hod}}^1(X, \text{GL}_n(\mathbb{C}))) := H_{\text{Del}}^1(X, \text{GL}_n(\mathbb{C}))|_{\mathbb{P}^1 \setminus \{\infty\}}$ as the complex conjugate of the Hodge filtration. In particular the Dolbeault space $H_{\text{Dol}}^1(X, \text{GL}_n(\mathbb{C}))$ should be thought of as the associated graded of the nonabelian Hodge filtration. The nonabelian counterpart of the usual property that a smooth projective X has a Hodge filtration concentrated in positive degrees is the statement [Simpson91, Lemma 16] that for any point $z \in H_{\text{Hod}}^1(X, \text{GL}_n(\mathbb{C}))$ the limit $\lim_{t \rightarrow 0} tz$ exists in $H_{\text{Hod}}^1(X, \text{GL}_n(\mathbb{C}))$.

4 The Gauss-Manin connection

4.1 Variations of geometric origin

A(vii) The most important example of \mathbb{Z} -variations of Hodge structures are the variations of geometric origin which we review next in the simplest geometric situation.

Let $f : X \rightarrow S$ be a smooth projective morphism between quasi projective varieties. Put $H_{\text{DR}}^i(X/S, \mathbb{C})$ for the sheaf of relative de Rham cohomology of degree i . Due to our assumptions about f the sheaf $H_{\text{DR}}^i(X/S, \mathbb{C})$ is locally free and of finite rank, i.e. corresponds to a vector bundle which we will denote again by $H_{\text{DR}}^i(X/S, \mathbb{C})$.

Algebraically the bundle $H_{\text{DR}}^i(X/S, \mathbb{C})$ is constructed from the relative de Rham complex on X . To see how this works recall first the following standard notation.

Given an abelian category \mathcal{A} , a complex K^\bullet of objects in \mathcal{A} , and an integer n one defines the *stupid* truncation $\sigma_{\leq n} K^\bullet$ of K^\bullet as the sub complex with terms

$$(\sigma_{\geq n} K^\bullet)^i := \begin{cases} 0 & \text{when } i < n \\ K^i & \text{when } i \geq n \end{cases}$$

Define also $\sigma_{> n} K^\bullet := \sigma_{\geq (n+1)} K^\bullet$ and $\sigma_{\leq n} K^\bullet = K^\bullet / \sigma_{< n} K^\bullet$.

Denote by Ω_f^\bullet the full de Rham complex of relative differential forms on X/S and consider the truncations $\sigma_{< p} \Omega_f^\bullet$ and $\sigma_{\geq p} \Omega_f^\bullet$. Explicitly $\sigma_{< p} \Omega_f^\bullet$ is the complex

$$\mathcal{O}_X \rightarrow \Omega_f^1 \rightarrow \dots \rightarrow \Omega_f^{p-1}$$

(\mathcal{O}_X is in degree zero) and $\sigma_{\geq p}\Omega_f^\bullet$ is the complex

$$\dots \rightarrow 0 \rightarrow \Omega_f^p \rightarrow \Omega_f^{p+1} \rightarrow \dots$$

(Ω_f^p is in degree p). One has $H_{\text{DR}}^i(X/S, \mathbb{C}) = \mathbb{R}^i f_* \Omega_f^\bullet$ since the de Rham complex resolves the constant sheaf \mathbb{C} . Furthermore the degeneration of the Hodge-de Rham spectral sequence gives $F^p H_{\text{DR}}^i(X/S, \mathbb{C}) = \mathbb{R}^i f_* \sigma_{\geq p} \Omega_f^\bullet$ and $H_{\text{DR}}^i(X/S, \mathbb{C})/F^p H_{\text{DR}}^i(X/S, \mathbb{C}) = \mathbb{R}^i f_* \sigma_{< p} \Omega_f^\bullet$, [Deligne68], [Deligne72]. By the universal coefficients theorem the image of $R^i f_* \mathbb{Z}$ in $H_{\text{DR}}^i(X/S, \mathbb{C})$ is a full lattice and therefore by the covering homotopy property of the cover $R^i f_* \mathbb{Z} \rightarrow S$ we get a canonical integrable connection on $H_{\text{DR}}^i(X/S, \mathbb{C})$ called the *Gauss-Manin connection*. It satisfies the Griffiths transversality condition and the first Chern class of any ample line bundle on X induces a horizontal Hermitian pairing that polarizes the variation $H_{\text{DR}}^i(X/S, \mathbb{C})$ [Griffiths69].

Even though the above topological definition of the Gauss-Manin connection is very intuitive, sometimes it is more convenient to have a cohomological description of the connection. Manin was the first one to realize that this connection can be defined in purely algebraic terms [Manin63]. Later Grothendieck [Grothendieck68] gave a universal algebraic construction of the connection which lead to the notion of a crystal. In general his construction gives a connection on the de Rham cohomology as an object in the derived category on S but in our simple case of a smooth projective $f : X \rightarrow S$ it admits the following explicit description [Katz-Oda68, Katz70].

Let Ω_X^\bullet be the global de Rham complex on X . Let I^1 denote the sub complex which is the image of $\Omega_X^\bullet \otimes_{\mathcal{O}_X} f^* \Omega_S^1$ and let I^2 denote the image of $\Omega_X^{\bullet-1} \otimes_{\mathcal{O}_X} f^* \Omega_S^1$. Observe that the relative de Rham complex is the quotient $\Omega_f^\bullet = \Omega_X^\bullet / I^1$ and that there is an isomorphism $I^1 / I^2 = \Omega_f^{\bullet-1} \otimes_{\mathcal{O}_X} f^* \Omega_S^1$. In particular we have an exact sequence of complexes

$$0 \longrightarrow \Omega_f^{\bullet-1} \otimes_{\mathcal{O}_X} f^* \Omega_S^1 \longrightarrow \Omega_X^\bullet / I^2 \longrightarrow \Omega_f^\bullet \longrightarrow 0.$$

The Gauss-Manin connection then is the first edge homomorphism of the hyper derived sequence, i.e. the map

$$d^{\text{GM}} : \mathbb{R}^i f_* \Omega_f^\bullet \longrightarrow \mathbb{R}^{i+1} f_* \Omega_f^\bullet[-1] \otimes \Omega_S^1 = \mathbb{R}^i f_* \Omega_f^\bullet \otimes \Omega_S^1.$$

NA(vii) Suppose again that $f : X \rightarrow S$ is a smooth projective morphism of quasi projective varieties and assume for simplicity that the fibers of f are connected. Denote by $\mathcal{M}_{\text{DR}}(X/S, n)$ the relative moduli stack of rank n local systems (see [Simpson94, Theorem 4.7] for existence). Simpson had shown [Simpson95] that $\mathcal{M}_{\text{DR}}(X/S, n)$ has a natural structure of a crystal of stacks over S dubbed by him the *nonabelian Gauss-Manin connection*. The construction mimics Grothendieck's definition for the abelian case in the moduli context. What makes the construction work is the observation that if S' is an S -scheme which contains a closed subscheme S'_0 defined by a nilpotent ideal then a crystal on X'/S' is

canonically equivalent to a crystal on X'_0/S' . Here $X' = X \times_S S'$ and $X'_0 = X \times_S S'_0$. Because of that the functor $M_{\text{crys}}^{\natural}(X/S, n)$ that assigns to a pair $S'_0 \subset S'$ the set of isomorphism classes of crystals of rank n vector bundles on X'_0/S' is a crystal of functors. Moreover Simpson proves [Simpson95, Lemmas 8.1 and 8.2] that $M_{\text{crys}}^{\natural}(X/S, n)$ is isomorphic to the relative de Rham moduli functor $M_{\text{DR}}^{\natural}(X/S, n)$. Since the latter is represented by $\mathcal{M}_{\text{DR}}(X/S, n)$ one gets a structure of a S -crystal on $\mathcal{M}_{\text{DR}}(X/S, n)$, i.e. an isomorphism

$$\varphi^{\text{GM}} : p_1^* \mathcal{M}_{\text{DR}}(X/S, n) \xrightarrow{\sim} p_2^* \mathcal{M}_{\text{DR}}(X/S, n)$$

on $(S \times S)^\wedge$ satisfying the usual cocycle condition. The same construction works also for the relative moduli space $H_{\text{DR}}^1(X/S, \text{GL}_n(\mathbb{C}))$ since the moduli functor $M_{\text{DR}}^{\natural}(X/S, n)$ is universally coarsely represented by the scheme $H_{\text{DR}}^1(X/S, \text{GL}_n(\mathbb{C}))$. It turns out that the regular points of $\mathcal{M}_{\text{DR}}(X/S, n)$ (respectively $H_{\text{DR}}^1(X/S, \text{GL}_n(\mathbb{C}))$) form a local system of stacks (respectively schemes) on S . In the same way we may define the moduli stack $\mathcal{M}_{\text{Dol}}(X/S, n)$ of relative Higgs bundles of rank n .

It turns out that the stacks $\mathcal{M}_{\text{DR}}(X/S, n)$ and $\mathcal{M}_{\text{Dol}}(X/S, n)$ are better approximations of the first non-abelian de Rham and Dolbeault cohomology than the moduli schemes $H_{\text{DR}}^1(X/S, \text{GL}_n(\mathbb{C}))$ and $H_{\text{Dol}}^1(X/S, \text{GL}_n(\mathbb{C}))$. This can be seen in many ways and is especially transparent when one considers non-abelian cohomology of degree greater than one [Simpson99]. Another indication is the fact that the set of \mathbb{C} -valued points of say $\mathcal{M}_{\text{DR}}(X/S, n)$ is exactly the set of equivalence classes of relative local systems on X/S whereas the set of \mathbb{C} -points of $H_{\text{DR}}^1(X/S, \text{GL}_n(\mathbb{C}))$ is the set of equivalence classes of semisimplifications of rank n relative local systems.

From now on we will freely work with these stacky non-abelian cohomologies.

Our first task describe the corresponding nonabelian connection explicitly at smooth points of the moduli. The natural framework for such a description is that of algebraic stacks and their tangent stacks. Our main references are [Laumon-Moret-Bailly92] and [Vistoli89, Appendix]. For the convenience of the reader we have reproduced the necessary statements in Appendix A.

First we will need an intrinsic modular description of the tangent stacks (see Section A.2 for a definition) of the moduli stacks $\mathcal{M}_{\text{DR}}(X/S, n)$ and $\mathcal{M}_{\text{Dol}}(X/S, n)$. Denote by

$$\begin{aligned} \pi_{\text{DR}} : \mathcal{M}_{\text{DR}}(X/S, n) &\longrightarrow S \\ \pi_{\text{Dol}} : \mathcal{M}_{\text{Dol}}(X/S, n) &\longrightarrow S \end{aligned}$$

the structure morphisms of the relative de Rham and Dolbeault stacks respectively. Let $(T; F, \nabla) \in \text{Ob}(\mathcal{M}_{\text{DR}}(X/S, n))$ and $(T; E, \theta) \in \text{Ob}(\mathcal{M}_{\text{Dol}}(X/S, n))$. In other words $T \rightarrow S$ is an S -scheme and if $X_T := X \times_S T$, then (F, ∇) and (E, θ) are a relative local system and a relative Higgs bundle on X_T/T . As usual we will describe the vertical and the total tangent stacks of the de Rham and Dolbeault stacks in terms of suitable deformation-obstruction complexes.

Lemma-Definition 4.1 *In the above notation put $f_T : X_T \rightarrow T$ for the natural projection. Then there are well defined complexes of sheaves on X_T*

De Rham version $\mathfrak{g}_{f_T}(F, \nabla) := (\text{End}(F) \otimes \Omega_{f_T}^\bullet, \text{ad}_\nabla)$, where

$$\text{ad}_\nabla(m) = [d^\nabla, m] = d^\nabla \circ m - (m \otimes \text{id}) \circ d^\nabla.$$

Dolbeault version $\mathfrak{g}_{f_T}(E, \theta) := (\text{End}(E) \otimes \Omega_{f_T}^\bullet, \text{ad}_\theta)$, where

$$\text{ad}_\theta(m) = [\theta, m] = \theta \circ m - (m \otimes \text{id}) \circ \theta.$$

Proof. The only thing that needs checking is the \mathcal{O}_{X_T} -linearity of $\text{ad}_\nabla(m) : F \rightarrow F \otimes \Omega_{f_T}^1$ and $\text{ad}_\theta(m) : E \rightarrow E \otimes \Omega_{f_T}^1$ respectively.

Since both m and θ are \mathcal{O}_{X_T} -linear it is clear that $\text{ad}_\theta(m)$ will also be \mathcal{O}_{X_T} -linear. To check the de Rham case we need to show that for any $m \in \text{End}(F)$ and any $a \in F$, $\phi \in \mathcal{O}_{X_T}$ we have $\text{ad}_\nabla(m)(\phi a) = \phi \text{ad}_\nabla(m)(a)$. Put $d_{f_T} : \Omega_{f_T}^i \rightarrow \Omega_{f_T}^{i+1}$ for the exterior differentiation along the fibers of f_T . By definition we have

$$\begin{aligned} \text{ad}_\nabla(m)(\phi a) &= [d^\nabla, m](\phi a) = d^\nabla(m(\phi a)) - (m \otimes \text{id})(d^\nabla(\phi a)) \\ &= d^\nabla(\phi \cdot m(a)) - (m \otimes \text{id})(a \otimes d_{f_T} \phi + \phi \cdot d^\nabla(a)) \\ &= m(a) \otimes d_{f_T} \phi + \phi d^\nabla \circ m(a) - m(a) \otimes d_{f_T} \phi + \phi(m \otimes \text{id}) \circ d^\nabla(a) \\ &= \phi \text{ad}_\nabla(m)(a), \end{aligned}$$

which proves the lemma since the integrability of ∇ and the symmetry condition $\theta \wedge \theta = 0$ on θ guarantee that ad_∇ and ad_θ will be differentials. \square

The complexes of the previous lemma carry information about the vertical tangent stacks of the structure morphisms π_{DR} and π_{Dol} . Before we explain that, recall the following construction. Given any algebraic S -stack \mathcal{X} and any complex of sheaves of abelian groups $E^0 \rightarrow E^1$ on \mathcal{X} one may consider the stack theoretic quotient of the translation action of E^0 on E^1 . In this way we get an S -stack $h^1/h^0(E^\bullet) := [E^1/E^0]$ having also a structure of a strictly commutative Picard stack over \mathcal{X} (see [SGA4, Section 1.4 of Exposé XVIII] and [Behrend-Fantechi97, Section 2] for details). For an object $V \in \text{Ob}(\mathcal{X})$ the groupoid of sections of $h^1/h^0(E^\bullet)$ over V is the category of pairs (R, r) , where R is an E^0 -torsor on V and $r : R \rightarrow E_{|V}^1$ is an E^0 -equivariant morphism of sheaves on V .

Denote by $\mathcal{M}_{\text{DR}}^o(X/S, n)$ and $\mathcal{M}_{\text{Dol}}^o(X/S, n)$ the parts of the stacks $\mathcal{M}_{\text{DR}}(X/S, n)$ and $\mathcal{M}_{\text{Dol}}(X/S, n)$ over which the morphisms π_{DR} and π_{Dol} are smooth. Put $(F_{\text{un}}, \nabla_{\text{un}}) \rightarrow \mathcal{M}_{\text{DR}}(X/S, n) \times_S X$ and $(E_{\text{un}}, \theta_{\text{un}}) \rightarrow \mathcal{M}_{\text{Dol}}(X/S, n) \times_S X$ for the universal families and let $f_{\text{DR}} : \mathcal{M}_{\text{DR}}(X/S, n) \times_S X \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$ and $f_{\text{Dol}} : \mathcal{M}_{\text{Dol}}(X/S, n) \times_S X \rightarrow \mathcal{M}_{\text{Dol}}(X/S, n)$ denote the natural projections. We have the following lemma.

Lemma 4.2

- (a) *The complexes $\mathfrak{g}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}})$ and $\mathfrak{g}_{f_{\text{Dol}}}(E_{\text{un}}, \theta_{\text{un}})$ are the deformation-obstruction complexes for the moduli functors $M_{\text{DR}}^\natural(X/S, n)$ and $M_{\text{Dol}}^\natural(X/S, n)$ over S . That is, the infinitesimal automorphisms, deformations and obstructions of the pairs (F, ∇) and (E, θ) on the fixed X_T are parameterized respectively by the vector spaces*

geometric objects	(F, ∇) with $T \rightarrow S$ kept fixed	(E, θ) with $T \rightarrow S$ kept fixed
infinitesimal automorphisms	$\mathbb{H}^0(X_T, \mathfrak{g}_{f_T}(F, \nabla))$	$\mathbb{H}^0(X_T, \mathfrak{g}_{f_T}(E, \theta))$
infinitesimal deformations	$\mathbb{H}^1(X_T, \mathfrak{g}_{f_T}(F, \nabla))$	$\mathbb{H}^1(X_T, \mathfrak{g}_{f_T}(E, \theta))$
infinitesimal obstructions	$\mathbb{H}^2(X_T, \mathfrak{g}_{f_T}(F, \nabla))$	$\mathbb{H}^2(X_T, \mathfrak{g}_{f_T}(E, \theta))$

(b) *There are isomorphisms of Picard stacks over $\mathcal{M}_{\text{DR}}^o(X/S, n)$ and $\mathcal{M}_{\text{Dol}}^o(X/S, n)$ respectively:*

$$\begin{aligned}
T_{\pi_{\text{DR}}} &= h^1/h^0(\mathbb{R}^0 f_{\text{DR}*} \mathfrak{g}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}}) \xrightarrow{0} \mathbb{R}^1 f_{\text{DR}*} \mathfrak{g}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}})) \\
T_{\pi_{\text{Dol}}} &= h^1/h^0(\mathbb{R}^0 f_{\text{Dol}*} \mathfrak{g}_{f_{\text{Dol}}}(E_{\text{un}}, \theta_{\text{un}}) \xrightarrow{0} \mathbb{R}^1 f_{\text{Dol}*} \mathfrak{g}_{f_{\text{Dol}}}(E_{\text{un}}, \theta_{\text{un}}))
\end{aligned}$$

Proof. (a) The proof is an easy but lengthy cocycle computation. We will work it out for the infinitesimal deformations of de Rham case only. The other cases are completely analogous. Put B for the spectrum of the dual numbers, i.e. B is the local nilpotent scheme $\text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$. Suppose we are given an infinitesimal deformation $(\tilde{F}, \tilde{\nabla})$ of (F, ∇) . Concretely $\tilde{F} \rightarrow X_T \times B$ is a locally free sheaf of rank n that specializes to F over the closed point of B and $\tilde{\nabla}$ is a relative integrable connection on \tilde{F} which specializes to ∇ . Choose an acyclic Čech cover \mathfrak{U} of X_T by Zariski open sets which trivializes the bundle F and denote by $h_U : F|_U \rightarrow \mathcal{O}_U^{\oplus n}$ a trivialization of F over $U \in \mathfrak{U}$. In terms of the cover \mathfrak{U} the bundle F is described by the cocycle $\{g_{UV}\} \in Z^1(\mathfrak{U}, \text{GL}_n(\mathcal{O}_{X_T}))$ where $g_{UV} = h_U \circ h_V^{-1}$. Similarly the connection ∇ is given by the cochain $\{a_U\} \in C^0(\mathfrak{U}, \text{End}(\mathcal{O}_{X_T}) \otimes \Omega_{f_T}^1)$ where $a_U = (h_U \otimes \text{id}_{\Omega^1}) \circ d^{\nabla} \circ h_U^{-1} - d$. Since $X_T \times B$ is a nilpotent scheme whose reduced support is X_T we can describe the pair $(\tilde{F}, \tilde{\nabla})$ in terms of the covering \mathfrak{U} by a pair of cochains $\{\tilde{g}_{UV}\} \in C^1(\mathfrak{U}, \text{GL}_n(\mathcal{O}_{X_T}[\varepsilon]/(\varepsilon^2)))$ and $\{\tilde{a}_U\} \in C^0(\mathfrak{U}, \text{End}(\mathcal{O}_{X_T}^{\oplus n}[\varepsilon]/(\varepsilon^2)))$ satisfying

$$(4.1.4) \quad \tilde{g}_{UV} = \tilde{g}_{UV} \tilde{g}_{VW}$$

$$(4.1.5) \quad \tilde{a}_U = \tilde{g}_{UV} d \tilde{g}_{VU} + \tilde{g}_{UV} \tilde{a}_V \tilde{g}_{VU}.$$

Also since $(\tilde{F}, \tilde{\nabla})$ specializes to (F, ∇) we have

$$\begin{aligned}
\tilde{g}_{UV} &= g_{UV} + \varepsilon x_{UV}, \\
\tilde{a}_U &= a_U + \varepsilon m_U,
\end{aligned}$$

with $\{x_{UV}\} \in C^1(\mathfrak{U}, \text{End}(\mathcal{O}_{X_T}^{\oplus n}))$ and $\{m_U\} \in C^0(\mathfrak{U}, \text{End}(\mathcal{O}_{X_T}^{\oplus n}) \otimes \Omega_{f_T}^1)$.

The integrability condition on the relative connection $\tilde{\nabla}$ is

$$d\tilde{a}_U + \frac{1}{2}[\tilde{a}_U, \tilde{a}_U] = 0.$$

with $d = d_{f_T}$ being the exterior differentiation along the fibers of f_T . Due to the integrability of ∇ this is equivalent to

$$(4.1.6) \quad dm_U + [a_U, m_U] = 0$$

By comparing the coefficients in front of ε in the cocycle condition (4.1.4) we obtain the identity $x_{UW} = g_{UV}x_{VW} + x_{UV}g_{VW}$ which can be rewritten as the identity $\mathfrak{x}_{UW} = \mathfrak{x}_{UV} + \mathfrak{x}_{VW}$ for $\mathfrak{x}_{UV} := h_U^{-1} \circ x_{UV} \circ h_V \in \Gamma(U \cap V, \text{End}(F))$. Thus $\{\mathfrak{x}_{UV}\} \in Z^1(\mathfrak{U}, \text{End}(F))$. Similarly the connection condition (4.1.5) gives

$$m_U = g_{UV}dx_{VU} + x_{UV}dg_{VU} + x_{UV}a_Vg_{VU} + g_{UV}m_Vg_{VU} + g_{UV}a_Vx_{VU}.$$

The next step is to rewrite dx_{VU} and dg_{VU} via Cartan homotopy formula:

$$\begin{aligned} m_U &= g_{UV} \circ d \circ x_{VU} + g_{UV} \circ x_{VU} \circ d + x_{UV} \circ d \circ g_{VU} + x_{UV} \circ g_{VU} \circ d \\ &\quad + x_{UV}a_Vg_{VU} + g_{UV}m_Vg_{VU} + g_{UV}a_Vx_{VU}. \end{aligned}$$

Taking into account that $g_{UV} = h_U \circ h_V^{-1}$, $\mathfrak{x}_{UV} = -\mathfrak{x}_{VU}$ and $(d^\nabla)_{|U \cap V} = (h_U \otimes \text{id}) \circ (d + a_U) \circ h_U^{-1} = (h_V \otimes \text{id}) \circ (d + a_V) \circ h_V^{-1}$ the last identity can be rewritten as

$$\mathfrak{m}_U - \mathfrak{m}_V = d^\nabla \circ \mathfrak{x}_{UV} - (\mathfrak{x}_{UV} \otimes \text{id}_{\Omega^1}) \circ d^\nabla$$

where $\mathfrak{m}_U = (h_U \otimes \text{id}) \circ m_U \circ h_U^{-1}$. In combination with (4.1.6) this implies that the pair $(\{\mathfrak{x}_{UV}\}, \{\mathfrak{m}_U\}) \in C^1(\mathfrak{U}, \mathfrak{g}_{f_T}(F, \nabla))$ is a cocycle and by construction completely determines the infinitesimal deformation $(\tilde{F}, \tilde{\nabla})$. It is straightforward to check that cohomologous cocycles correspond to isomorphic deformations and so $(\tilde{F}, \tilde{\nabla})$ determines and is determined by an element in $\mathbb{H}^1(X_T, \mathfrak{g}_{f_T}(F, \nabla))$.

(b) A cheap way to prove this part is to combine Simpson's formality result [Simpson92, Lemma 3.5] [Simpson95, Proposition 10.5] with the isomorphism (A.2.15).

One can also give a direct argument which we proceed to explain in the de Rham case. Fix an object $(T; F, \nabla) \in \text{Ob}(\mathcal{M}_{\text{DR}}(X/S, n))$ and let $\alpha_T : T \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$ be the corresponding morphism of S -stacks. The fiber stack $T_{\pi_{\text{DR}}, (T; F, \nabla)} := T_{\pi_{\text{DR}}} \times_{\alpha_T} \mathcal{M}_{\text{DR}}(X/S, n)$ is naturally a T -stack. Explicitly for a given T -scheme $U \rightarrow T$ let $X_U := X_T \times_T U$ and let $f_U : X_U \rightarrow U$ denote the natural projection. We abuse notation and write (F, ∇) for the pull-back of (F, ∇) to X_U . The groupoid of sections $T_{\pi_{\text{DR}}, (T; F, \nabla)}$ over U is the category of S -relative connections $(\tilde{F}, \tilde{\nabla}) \rightarrow X_U \times B$ such that if $j : X_U \hookrightarrow X_U \times B$ is the inclusion coming from the inclusion of the closed point in B , then there exists an isomorphism $j^*(\tilde{F}, \tilde{\nabla}) \cong (F_U, \nabla_U)$. The choice of such an isomorphism however is not considered part of the data $(\tilde{F}, \tilde{\nabla})$. Similarly, if we denote by $h^1/h_{\text{DR}, (T; F, \nabla)}^0$ the fiber-product

$$h^1/h^0(\mathbb{R}^0 f_{\text{DR}*} \mathfrak{g}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}}) \xrightarrow{0} \mathbb{R}^1 f_{\text{DR}*} \mathfrak{g}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}})) \times_{\alpha_T} \mathcal{M}_{\text{DR}}(X/S, n)$$

we can identify the groupoid of sections of $h^1/h_{\text{DR}, (T; F, \nabla)}^0$ over U as the category of all pairs (R, r) where R is an $\mathbb{H}^0(X_U, \mathfrak{g}_{f_U}(F, \nabla))$ torsor and $r \in \mathbb{H}^1(X_U, \mathfrak{g}_{f_U}(F, \nabla))$. Next we have a functor

$$a : T_{\pi_{\text{DR}}, (T; F, \nabla)} \longrightarrow h^1/h_{\text{DR}, (T; F, \nabla)}^0$$

given by $(\tilde{F}, \tilde{\nabla}) \mapsto (R_{(\tilde{F}, \tilde{\nabla})}, r_{(\tilde{F}, \tilde{\nabla})})$ where $R_{(\tilde{F}, \tilde{\nabla})}$ is the set of all surjective morphisms of crystals $(\tilde{F}, \tilde{\nabla}) \rightarrow (F, \nabla)$ on $X_T \times B$ and $r_{(\tilde{F}, \tilde{\nabla})}$ is the (Kodaira-Spencer) class of $(\tilde{F}, \tilde{\nabla})$ constructed in (a). Equivalently $R_{(\tilde{F}, \tilde{\nabla})}$ can be thought of as the set of all realizations of the $\Omega_{f_T}^\bullet$ -dg-module $(\tilde{F} \otimes \Omega_{f_T}^\bullet, d^{\tilde{\nabla}})$ as an extension of F by the $\Omega_{f_T}^\bullet$ -dg-module $(F \otimes \Omega_{f_T}^\bullet, d^\nabla)$ and $r_{(\tilde{F}, \tilde{\nabla})}$ as the extension class of any such realization. \square

To describe the full tangent spaces to the relative de Rham and Dolbeault stacks we will need to extend the complexes $\mathfrak{g}_{f_T}(F, \nabla)$ and $\mathfrak{g}_{f_T}(E, \theta)$ in a suitable fashion. For a coherent sheaf $F \rightarrow X_T$ denote by $\mathcal{E}_{f_T}(F)$ the f_T -relative Atiyah algebra of F . Then we have

Lemma-Definition 4.3 *For any F algebraic vector bundle on X_T one has:*

(i) *The natural morphism*

$$\begin{aligned} \ell : \quad \mathcal{E}_{f_T}(F) &\longrightarrow \mathcal{E}_{f_T}(F \otimes \Omega_{f_T}^1) \\ \partial &\longrightarrow \partial \otimes \text{id} + \text{id} \otimes L_{\sigma(\partial)} \end{aligned}$$

is well defined and \mathcal{O}_{X_T} -linear.

(ii) *There are well defined complexes on Y*

De Rham version

$$\mathcal{E}_{f_T}(F, \nabla) := \mathcal{E}_{f_T}(F) \xrightarrow{\text{ad}_\nabla} \text{End}(F) \otimes \Omega_{f_T}^1 \xrightarrow{\text{ad}_\nabla} \text{End}(F) \otimes \Omega_{f_T}^2 \rightarrow \dots$$

where $\text{ad}_\nabla(\partial) := d^\nabla \circ \partial - \ell(\partial) \circ d^\nabla$ for $\partial \in \mathcal{E}_{f_T}(F)$ and is as in lemma-definition 4.1 for $m \in \text{End}(F) \otimes \Omega_Y^i$.

Dolbeault version

$$\mathcal{E}_{f_T}(E, \theta) := \mathcal{E}_{f_T}(E) \xrightarrow{\text{ad}_\theta} \text{End}(E) \otimes \Omega_{f_T}^1 \xrightarrow{\text{ad}_\theta} \text{End}(E) \otimes \Omega_{f_T}^2 \rightarrow \dots$$

where $\text{ad}_\theta(\partial) := \theta \circ \partial - \ell(\partial) \circ \theta$ for $\partial \in \mathcal{E}_{f_T}(E)$ and is as in lemma-definition 4.1 for $m \in \text{End}(E) \otimes \Omega_{f_T}^i$.

Proof. To prove (i) we first need to show that for any local section $\partial \in \mathcal{E}_{f_T}(F)$ the \mathbb{C} -linear map $\ell(\partial) : F \otimes \Omega_{f_T}^1 \rightarrow F \otimes \Omega_{f_T}^1$ belongs to the Atiyah algebra of F . For any local sections $\varphi \in \mathcal{O}_{X_T}$, $a \in F$, $\alpha \in \Omega_{f_T}^1$ we have by definition

$$\begin{aligned} \ell(\partial)((\varphi a) \otimes \alpha) &= \partial(\varphi a) \otimes \alpha + (\varphi a) \otimes L_{\sigma(\partial)}\alpha = \\ &= (L_{\sigma(\partial)}\varphi)a \otimes \alpha + \varphi \partial a \otimes \alpha + \varphi a \otimes L_{\sigma(\partial)}\alpha = \\ &= \varphi \partial a \otimes \alpha + a (\varphi L_{\sigma(\partial)}\alpha + (L_{\sigma(\partial)}\varphi)\alpha) = \\ &= \partial a \otimes (\varphi \alpha) + a \otimes L_{\sigma(\partial)}(\varphi \alpha) = \ell(\partial)(a \otimes \varphi \alpha). \end{aligned}$$

In particular $\ell(\partial)$ is a well defined \mathbb{C} -linear endomorphism of $F \otimes \Omega_{f_T}^1$ since it respects the \mathcal{O}_{X_T} -linearity of the tensor product of F and $\Omega_{f_T}^1$. Moreover the second line in the above identity shows that for any $A \in F \otimes \Omega_{f_T}^1$ we have $\ell(\partial)(\varphi A) = \varphi \ell(\partial)(A) + (L_{\sigma(\partial)}\varphi)A$ and thus $\ell(\partial)$ is actually an element in $\mathcal{E}(F \otimes \Omega_{f_T}^1)$ with $\sigma(\ell(\partial)) = \sigma(\partial)$. Since the \mathcal{O}_{X_T} -linearity of ℓ is clear from its definition this completes the proof of (i).

For part (ii) we will only check that $\text{ad}_\nabla(\partial)$ is \mathcal{O}_{X_T} -linear for every $\partial \in \mathcal{E}_{f_T}(F)$ since the corresponding statement for ad_θ is checked in exactly the same way. Let a and φ be as before. Then

$$\begin{aligned} \text{ad}_\nabla(\partial)(\varphi a) &= (d^\nabla \circ \partial - \ell(\partial) \circ d^\nabla)(\varphi a) = \\ &= d^\nabla \circ ((L_{\sigma(\partial)}\varphi)a + \varphi(\partial a)) - \ell(\partial)(a \otimes d\varphi + \varphi d^\nabla a) = \\ &= a \otimes (d \circ L_{\sigma(\partial)})\varphi + \varphi(d^\nabla \circ \partial)a \\ &\quad - a \otimes (L_{\sigma(\partial)} \circ d)\varphi - \varphi(\ell(\partial) \circ d^\nabla)a. \end{aligned}$$

Also for a vertical vector field $v \in T_{f_T}$ the Cartan homotopy formula gives

$$L_v = d \circ i_v + i_v \circ d \text{ on } \Omega_{f_T}^\bullet,$$

where i_v is the contraction with v .

Therefore $d \circ L_v = d \circ i_v \circ d = L_v \circ d$ and hence

$$\text{ad}_\nabla(\partial)(\varphi a) = \varphi \text{ad}_\nabla(\partial)(a)$$

which completes the proof of the lemma. \square

Again the complexes $\mathcal{E}_{f_T}(F, \nabla)$ and $\mathcal{E}_{f_T}(E, \theta)$ can be interpreted as deformation-obstruction complexes but before we state the result we need to recall some standard notation.

Given an abelian category \mathcal{A} , a complex K^\bullet of objects in \mathcal{A} , and an integer n one defines the *canonical* truncation $\tau_{\leq n} K^\bullet$ of K^\bullet as the sub complex with terms

$$(\tau_{\leq n} K^\bullet)^i := \begin{cases} K^i & \text{when } i < n \\ \ker(d) & \text{when } i = n \\ 0 & \text{when } i > n \end{cases}$$

Define also $\tau_{< n} K^\bullet := \tau_{\leq (n-1)} K^\bullet$ and $\tau_{\geq n} K^\bullet = K^\bullet / \tau_{< n} K^\bullet$. By definition we have $H^i \tau_{\leq n} K^\bullet = H^i K^\bullet$ when $i \leq n$ and $H^i \tau_{\leq n} K^\bullet = 0$ for $i > n$. Similarly $H^i \tau_{\geq n} K^\bullet = H^i K^\bullet$ when $i \geq n$ and $H^i \tau_{\geq n} K^\bullet = 0$ for $i < n$.

We are now ready to state

Lemma 4.4

- (a) *The complexes $\mathcal{E}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}})$ and $\mathcal{E}_{f_{\text{Dol}}}(E_{\text{un}}, \theta_{\text{un}})$ are the deformation-obstruction complexes for the moduli functors $M_{\text{DR}}^\natural(X/S, n)$ and $M_{\text{Dol}}^\natural(X/S, n)$. That is, the infinitesimal automorphisms, deformations and obstructions of the triples (X_T, F, ∇) and (X_T, E, θ) over S are parameterized respectively by the vector spaces*

geometric objects	(X_T, F, ∇) over S	(X_T, E, θ) over S
infinitesimal automorphisms	$\mathbb{H}^0(X_T, \mathcal{E}_{f_T}(F, \nabla))$	$\mathbb{H}^0(X_T, \mathcal{E}_{f_T}(E, \theta))$
infinitesimal deformations	$\mathbb{H}^1(X_T, \mathcal{E}_{f_T}(F, \nabla))$	$\mathbb{H}^1(X_T, \mathcal{E}_{f_T}(E, \theta))$
infinitesimal obstructions	$\mathbb{H}^2(X_T, \mathcal{E}_{f_T}(F, \nabla))$	$\mathbb{H}^2(X_T, \mathcal{E}_{f_T}(E, \theta))$

(b) *There are isomorphisms of Picard stacks over $\mathcal{M}_{\text{DR}}^o(X/S, n)$ and $\mathcal{M}_{\text{Dol}}^o(X/S, n)$ respectively:*

$$\begin{aligned} T_{\pi_{\text{DR}}} &= h^1/h^0(\tau_{\leq 1} \mathbb{R}f_{\text{DR}*} \mathcal{E}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}})) \\ T_{\pi_{\text{Dol}}} &= h^1/h^0(\tau_{\leq 1} \mathbb{R}f_{\text{Dol}*} \mathcal{E}_{f_{\text{Dol}}}(E_{\text{un}}, \theta_{\text{un}})) \end{aligned}$$

Proof. Again we will give a proof only for the de Rham case. Also we will only consider the case when $T = \{s\} \rightarrow S$ is a closed point. The proof carries over verbatim to the case of a general $T \rightarrow S$ and is left to the reader. To simplify notation put $Y := X_T = X_s$.

Let as before $B = \text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ and suppose we are given a deformation $(\tilde{Y}, \tilde{F}, \tilde{\nabla})$ of (Y, F, ∇) over B . Choose again an acyclic Čech covering \mathfrak{U} of Y consisting of affine open sets over which F trivializes.

The structure sheaf of \tilde{Y} is an extension of \mathcal{O}_Y by an ideal of square zero which is isomorphic to \mathcal{O}_Y as an \mathcal{O}_Y -module. Since Y is smooth this extension will split over any affine open subset due to the infinitesimal lifting property. Therefore for every $U \in \mathfrak{U}$ we can choose a ring isomorphism $c_U : \mathcal{O}_{\tilde{Y}|U} \rightarrow \mathcal{O}_U \oplus \mathcal{O}_U$ with the ring structure on $\mathcal{O}_U \oplus \mathcal{O}_U$ given by $(f, a) \cdot (g, b) := (fg, fb + ga)$. Thus the sheaf of rings $\mathcal{O}_{\tilde{Y}}$ on the topological space Y is described by the 1-cocycle on the nerve of \mathfrak{U} given by $D_{UV} := c_U \circ c_V^{-1}$. Clearly $D_{UV} \in \Gamma(U \cap V, \text{End}_{\mathbb{C}}(\mathcal{O}_Y \oplus \mathcal{O}_Y))$ and since D_{UV} is an isomorphism of the corresponding split extensions we can write it in the form

$$D_{UV} = \begin{pmatrix} 1 & 0 \\ \partial_{UV} & 1 \end{pmatrix},$$

where ∂_{UV} is some \mathbb{C} -linear homomorphism from the first copy of $\mathcal{O}_{U \cap V}$ to the second one. Also D_{UV} has to be a ring automorphism of $\mathcal{O}_{U \cap V} \oplus \mathcal{O}_{U \cap V}$ for the ring structure described above. This is easily seen to be equivalent to $\partial_{UV} \in \Gamma(U \cap V, \text{Der}(\mathcal{O}_Y))$ and hence the sheaf of rings $\mathcal{O}_{\tilde{Y}}$ is described by the cocycle $\{\partial_{UV}\} \in Z^1(\mathfrak{U}, T_Y)$ which represents the Kodaira-Spencer class of the infinitesimal deformation $\tilde{Y} \rightarrow B$ (see [EGA4, 0_{IV} 20] for more details).

Let h_U, g_{UV}, a_U be as in lemma 4.2. The triple $(\tilde{Y}, \tilde{F}, \tilde{\nabla})$ now is encoded in a pair of cochains $\{\tilde{g}_{UV}\} \in C^1(\mathfrak{U}, \text{GL}(n, \mathcal{O}_{\tilde{Y}}))$ and $\{\tilde{a}_U\} \in C^0(\mathfrak{U}, \text{End}(\mathcal{O}_{\tilde{Y}}^{\oplus n}))$ satisfying the same cocycle and connection conditions (4.1.4) and (4.1.5). Again we can write

$$\begin{aligned} \tilde{g}_{UV} &= g_{UV} + \varepsilon x_{UV}, \\ \tilde{a}_U &= a_U + \varepsilon m_U, \end{aligned}$$

$\{m_U\} \in C^0(\mathfrak{U}, \text{End}(\mathcal{O}_Y^{\oplus n}) \otimes \Omega_Y^1)$, but only this time $\{x_{UV}\} \in C^1(\mathfrak{U}, \text{End}_{\mathbb{C}}(\mathcal{O}_Y^{\oplus n}))$. Since the decomposition $\tilde{g}_{UV} = g_{UV} + \varepsilon x_{UV}$ comes from a trivialization $\tilde{F}|_U \cong \mathcal{O}_{\tilde{Y}|U}^{\oplus n} \xrightarrow{c_U} (\mathcal{O}_U \oplus \mathcal{O}_U)^{\oplus n}$ we have that $x_{UV} : \mathcal{O}_{U \cap V}^{\oplus n} \rightarrow \mathcal{O}_{U \cap V}^{\oplus n}$ is a differential operator of order ≤ 1 and that $x_{UV} - \partial_{UV} \otimes \text{id}_{\mathbb{C}^n}$ is a $\mathcal{O}_{U \cap V}$ -linear endomorphism of $\mathcal{O}_{U \cap V}^{\oplus n}$. In other words, the Kodaira-Spencer class of the deformation \tilde{Y} can be recovered from \tilde{g}_{UV} as the symbol of the differential operator x_{UV} . Thus $\mathfrak{x}_{UV} := h_U^{-1} \circ x_{UV} \circ h_V$ is a section of $\mathcal{E}(F)$ and since the identity $x_{UW} = g_{UV}x_{VW} + x_{UV}g_{VW}$ is equivalent to $\mathfrak{x}_{UW} = \mathfrak{x}_{UV} + \mathfrak{x}_{VW}$ we have $\{\mathfrak{x}_{UV}\} \in Z^1(\mathfrak{U}, \mathcal{E}(F))$. Rewriting again the connection condition (4.1.5) in terms of $\mathfrak{m}_U = (h_U \otimes \text{id}) \circ m_U \circ h_U^{-1}$ we get

$$\mathfrak{m}_U - \mathfrak{m}_V = d^\nabla \circ \mathfrak{x}_{UV} - \ell(\mathfrak{x}_{UV}) \circ d^\nabla.$$

Together with the integrability of ∇ this gives $(\mathfrak{x}_{UV}, \mathfrak{m}_{UV}) \in Z^1(\mathfrak{U}, \mathcal{E}_Y(F, \nabla))$ which completes the proof of part (a) of the lemma.

For part (b) we invoke (similarly to the proof of Lemma 4.2) the isomorphism (A.2.15). Note that unlike part (b) of Lemma 4.2 no formality is claimed here since already the deformation-obstruction complexes $\tau_{\leq 1} Rf_* T_f$ need not be formal. However if the fibers of f have no infinitesimal automorphisms the complex $\tau_{\leq 1} Rf_* T_f$ is automatically formal and we again have

$$\begin{aligned} \tau_{\leq 1} \mathbb{R}f_{\text{DR}*} \mathcal{E}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}}) &= [\mathbb{R}^0 f_{\text{DR}*} \mathcal{E}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}}) \xrightarrow{0} \mathbb{R}^1 f_{\text{DR}*} \mathcal{E}_{f_{\text{DR}}}(F_{\text{un}}, \nabla_{\text{un}})] \\ \tau_{\leq 1} \mathbb{R}f_{\text{Dol}*} \mathcal{E}_{f_{\text{Dol}}}(F_{\text{un}}, \nabla_{\text{un}}) &= [\mathbb{R}^0 f_{\text{Dol}*} \mathcal{E}_{f_{\text{Dol}}}(E_{\text{un}}, \theta_{\text{un}}) \xrightarrow{0} \mathbb{R}^1 f_{\text{Dol}*} \mathcal{E}_{f_{\text{Dol}}}(F_{\text{un}}, \nabla_{\text{un}})] \end{aligned}$$

due to Simpson formality result [Simpson92, Lemma 3.5] [Simpson95, Proposition 10.5]. \square

Next we use the information about the vertical and full tangent spaces of the fibration $\pi_{\text{DR}} : \mathcal{M}_{\text{DR}}(X/S, n) \rightarrow S$, that we have just obtained, to give a description of the non-abelian Gauss-Manin connection. From now on, until the end of this section, we will work only with the part $\mathcal{M}_{\text{DR}}^o(X/S, n)$ of $\mathcal{M}_{\text{DR}}(X/S, n)$ over which the morphism π_{DR} is smooth. Let $(Y; F, \nabla) \in \mathcal{M}_{\text{DR}}^o(X/S, n)$ be a closed point lying over $s \in S$. The non abelian Gauss-Manin connection is given by a map $GM_{(Y; F, \nabla)} : T_{S, s} \rightarrow T_{\mathcal{M}_{\text{DR}}^o(X/S, n), (Y; F, \nabla)}$ splitting the differential $d\pi_{\text{DR}} : T_{\mathcal{M}_{\text{DR}}^o(X/S, n), (Y; F, \nabla)} \rightarrow T_{S, s}$. On the other hand, we know from the proof of Lemma 4.4 that $d\pi_{\text{DR}}$ fits in the commutative diagram

$$\begin{array}{ccc} T_{\mathcal{M}_{\text{DR}}^o(X/S, n), (Y; F, \nabla)} & \xrightarrow{d\pi_{\text{DR}}} & T_{S, s} \\ \kappa_{(Y; F, \nabla)} \downarrow & & \downarrow \kappa_Y \\ h^1/h^0(\tau_{\leq 1} \mathbb{R}\Gamma(\mathcal{E}_Y(F, \nabla))) & \xrightarrow{R\Gamma(\sigma)} & h^1/h^0(\tau_{\leq 1} R\Gamma(T_Y)) \end{array}$$

where κ_Y and $\kappa_{(Y; F, \nabla)}$ denote the Kodaira-Spencer maps for Y and for the triple $(Y; F, \nabla)$ respectively and $R\Gamma(\sigma)$ is the map in cohomology induced by the symbol morphism of complexes $\sigma : \mathcal{E}_Y(F, \nabla) \rightarrow [T_Y \rightarrow 0 \rightarrow \dots]$. Recall next that since $\nabla : T_Y \rightarrow \mathcal{E}_Y(F)$ is a

connection, it splits the symbol map for F and that due to the integrability of ∇ we have $\text{ad}_\nabla \circ \nabla = 0$. Thus ∇ can be thought of as a morphism of complexes $[T_Y \rightarrow 0 \rightarrow \dots] \rightarrow \mathcal{E}_Y(F, \nabla)$ that splits the short exact sequence of complexes

$$0 \longrightarrow \mathfrak{g}_Y(F, \nabla) \longrightarrow \mathcal{E}_Y(F, \nabla) \xrightarrow{\sigma} [T_Y \rightarrow 0 \rightarrow \dots] \longrightarrow 0.$$

$\searrow \nabla$

Now the intrinsic description of the non-abelian Gauss-Manin connection is given by the following lemma.

Lemma 4.5 *The map $GM_{(Y;F,\nabla)}$ fits in the commutative diagram*

$$\begin{array}{ccc} T_{S,s} & \xrightarrow{GM_{(Y;F,\nabla)}} & T_{\mathcal{M}_{\text{DR}}^o(X/S,n),(Y;F,\nabla)} \\ \kappa_Y \downarrow & & \downarrow \kappa_{(Y;F,\nabla)} \\ h^1/h^0(R\Gamma(T_Y)) & \xrightarrow{R\Gamma(\nabla)} & h^1/h^0(\mathbb{R}\Gamma(\mathcal{E}_Y(F, \nabla))) \end{array}$$

Proof. Recall the definition of the map $GM_{(Y;F,\nabla)}$ (cf. **NA(vii)** and [Simpson95, Section 8]). Start with a tangent vector $v \in T_{S,s}$. Geometrically v corresponds to a morphism $i_v : B \rightarrow S$ mapping the closed point $o \in B$ to $s \in S$. In other words v gives a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & X^v \\ \downarrow & & \downarrow \\ o & \longrightarrow & B \end{array}$$

with $X^v := X \times_S B$ and j -the natural inclusion. Furthermore, since $\mathcal{M}_{\text{DR}}(X/S, n)$ represents the moduli functor of rank n local systems on X/S , the vector $GM_{(Y;F,\nabla)}(v) \in T_{\mathcal{M}_{\text{DR}}^o(X/S,n),(Y;F,\nabla)}$ can be interpreted as a relative rank n local system on X^v/B that restricts to (F, ∇) on Y . Now it remains only to note that specifying (F, ∇) is the same as specifying a structure $\varphi : p_1^* F \xrightarrow{\sim} p_2^* F$ of a crystal of vector bundles over Y and that j^* is an equivalence between the categories of crystals over X^v/B and crystals over Y/B [Grothendieck68, Appendix], [Simpson95, Proposition 8.4].

To make this explicit put $q^v : X^v \rightarrow (Y \times Y)^\wedge$ for the natural map and let $q^{v*} \nabla : F \rightarrow F \otimes j^* \Omega_{X^v/B}^1$ denote the map $e \mapsto j^*((p_2^* e - \varphi p_1^* e) \bmod J^2)$, where as usual J stands for the ideal sheaf of $Y \subset (Y \times Y)^\wedge$. If $U \subset Y$ is an affine open and $c_U : \mathcal{O}_{X^v|U} \rightarrow \mathcal{O}_U \oplus \mathcal{O}_U$ is as in Lemma 4.4, then c_U induces an isomorphism $\tilde{c}_U : \Omega_{X^v/B|U}^1 \rightarrow \Omega_U^1 \oplus \mathcal{O}_U$ that is uniquely characterized by the property $\tilde{c}_U \circ d \circ c_U^{-1}(f, a) = (df, a)$. Let \mathfrak{U} be a Čech covering of Y and let $\{\partial_{UV}\}$ be the \mathfrak{U} -cocycle representing $\kappa_Y(v)$. The sheaf $j^* \Omega_{X^v/B}^1$ is completely determined by the cocycle $\{\tilde{D}_{UV}\} \in Z^1(\mathfrak{U}, \text{End}(\Omega_Y^1 \oplus \mathcal{O}_Y))$, where $\tilde{D}_{UV}((\alpha, a)) = (\alpha, a + i_{\sigma(\partial_{UV})} \alpha)$ and the map $(d^{q^{v*} \nabla})|_U$ is given by the morphism $F_U \rightarrow (F_U \oplus F_U \otimes \Omega_U^1)$, $e \mapsto e \oplus \nabla e$.

Now, by what we said above, $\kappa_{(Y;F,\nabla)}(GM_{(Y;F,\nabla)}(v))$ is the Kodaira-Spencer class of the unique (up to isomorphism) relative local system (F^v, ∇^v) on X^v/B that restricts to $q^{v*} \nabla$.

Finally, the expression for $q^{v*}\nabla$ in terms of \mathfrak{U} and Lemma 4.4 show that $R\Gamma(\nabla)(v)$ restricts exactly to $q^{v*}\nabla$. \square

Suppose now we have an algebraic section $a : S \rightarrow \mathcal{M}_{\text{DR}}^o(X/S, n)$. We can use the previous lemma to express the property of a being horizontal for the non-abelian Gauss-Manin connection in geometric terms. Specifying a is the same as specifying a relative rank n local system on X/S , i.e. a vector bundle $F \rightarrow X$ of rank n and a relative integrable connection $d^{\nabla_f} : F \rightarrow F \otimes \Omega_f^1$. Denote by $\mathfrak{g}_f(F, \nabla_f)$ and $\mathcal{E}_f(F, \nabla_f)$ the relative versions of complexes in Lemma 4.2 and Lemma 4.4 respectively. As before, the relative connection ∇_f induces a morphism of complexes $\nabla_f : [T_f \rightarrow 0 \rightarrow \dots] \rightarrow \mathcal{E}_f(F, \nabla_f)$ and we have the following proposition.

Proposition 4.6 *The section a is horizontal with respect to the non-abelian Gauss-Manin connection on $\mathcal{M}_{\text{DR}}^o(X/S, n)$ if and only if one of the following equivalent conditions holds*

- (1) *There exists a global integrable connection $\nabla : T_X \rightarrow \mathcal{E}_X(F)$ which induces*

$$\nabla_f : T_f \rightarrow \mathcal{E}_f(F).$$

- (2) *The following diagram of objects in $D^b(S)$ commutes*

$$\begin{array}{ccc} T_S & \xrightarrow{\kappa_{(X/S; F, \nabla_f)}} & \tau_{\leq 0}(\mathbb{R}f_* \mathcal{E}_f(F, \nabla_f)[1]) \\ & \searrow \kappa_{X/S} & \nearrow \tau_{\leq 0}(Rf_* \nabla_f[1]) \\ & \tau_{\leq 0}(Rf_* T_f[1]) & \end{array}$$

Proof. The fact that (2) is equivalent to the horizontality of a is almost immediate. Indeed, according to Remark 2.13(iii), a is horizontal if and only if $da = GM$ as maps from T_S to $a^*T_{\mathcal{M}_{\text{DR}}^o(X/S, n)}$. In order to compare da and GM recall first that due to [SGA4, Proposition 1.4.15 of Exposé XVIII] the construction h^1/h^0 induces a 1-equivalence of the category of quasi-isomorphism classes of complexes of sheaves of amplitude one and the 2-category of Picard stacks. But by Lemma 4.4 we have $a^*T_{\mathcal{M}_{\text{DR}}^o(X/S, n)} = h^1/h^0(\tau_{\leq 1}(\mathbb{R}f_* \mathcal{E}_f(F, \nabla_f)))$ and so we can identify da with $\kappa_{(X/S; F, \nabla_f)}$. Similarly by Lemma 4.5 we have $GM = Rf_* \nabla_f \circ \kappa_{X/S}$.

Consequently, we need only to check the equivalence of (1) and (2). The short exact sequence of sheaves

$$0 \longrightarrow T_f \longrightarrow T_X \xrightarrow{df} f^{-1}T_S \longrightarrow 0,$$

pushes forward to a distinguished triangle in $D^b(S)$:

$$Rf_* T_f \longrightarrow Rf_* T_X \xrightarrow{df} Rf_* f^{-1}T_S \longrightarrow Rf_* T_f[1],$$

and the Kodaira-Spencer map $\kappa_{X/S} : T_S \rightarrow \tau_{\leq 0}(Rf_*T_f[1])$ is just the $\tau_{\leq 0}$ -truncation of the edge homomorphism $Rf_*f^{-1}T_S \rightarrow Rf_*T_f[1]$. Let $\kappa_{X/S}^1 : T_S \rightarrow R^1f_*T_f$ denote the naive Kodaira-Spencer map, i.e. the composition of $\kappa_{X/S}$ with the natural morphism

$$\tau_{\leq 0}(Rf_*T_f[1]) \rightarrow R^1f_*T_f$$

in $D^b(S)$.

In order to express $\kappa_{(X/S;F,\nabla_f)}$ as an edge homomorphism, we will just have to recast the local calculation from Lemma 4.4 into the global setting of the family $f : X \rightarrow S$. Following section 2.2 put $\mathcal{E}_X^\sim(F) \subset \mathcal{E}_X(F)$ for all differential operators whose symbol is in T_X^\sim . Since T_X^\sim centralizes T_f we have that $L_v(f^*\Omega_f^1) \subset f^*\Omega_f^1$ for all $v \in T_X^\sim$ and hence $L_{\sigma(\partial)}$ descends to a map in $\text{End}_{\mathbb{C}}(\Omega_f^1)$ for a $\partial \in \mathcal{E}_X^\sim(F)$. Now, the same line of reasoning as in Lemma-Definition 4.3 shows that the map $\tilde{\ell} : \mathcal{E}_X^\sim(F) \rightarrow \mathcal{E}_X^\sim(F \otimes \Omega_f^1)$ is well defined and that $\text{ad}_{\nabla_f}(\partial) := \nabla_f \circ \partial - \tilde{\ell}(\partial) \circ \nabla_f$ gives a complex of sheaves on X :

$$\mathcal{E}_X^\sim(F, \nabla_f) := [\mathcal{E}_X^\sim(F) \xrightarrow{\text{ad}_{\nabla_f}} \text{End}(F) \otimes \Omega_f^1 \xrightarrow{\text{ad}_{\nabla_f}} \text{End}(F) \otimes \Omega_f^2 \longrightarrow \dots].$$

Finally, due to the calculation in Lemma 4.4 we have that $\kappa_{(X/S;F,\nabla_f)}$ is nothing but the $\tau_{\leq 0}$ -truncation of the edge homomorphism for the distinguished triangle in $D^b(S)$:

$$\mathbb{R}f_*\mathcal{E}_f(F, \nabla_f) \longrightarrow \mathbb{R}f_*\mathcal{E}_X^\sim(F, \nabla_f) \xrightarrow{df} Rf_*f^{-1}T_S \longrightarrow \mathbb{R}f_*\mathcal{E}_f(F, \nabla_f)[1]$$

obtained as a push forward of the short exact sequence of complexes

$$0 \longrightarrow \mathcal{E}_f(F, \nabla_f) \longrightarrow \mathcal{E}_X^\sim(F, \nabla_f) \xrightarrow{df} f^{-1}T_S \longrightarrow 0.$$

Here, as usual, $f^{-1}T_S$ is thought of as a complex concentrated in degree zero. Again we put $\kappa_{(X/S;F,\nabla_f)}^1$ for the naive Kodaira-Spencer map given as the composition $T_S \rightarrow \tau_{\leq 0}(\mathbb{R}f_*\mathcal{E}_f(F, \nabla_f)[1]) \rightarrow \mathbb{R}^1f_*\mathcal{E}_f(F, \nabla_f)$.

Furthermore, the two complexes defining $\kappa_{X/S}$ and $\kappa_{(X/S;F,\nabla_f)}$ are tied up in the following commutative diagram

(4.1.7)

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \uparrow & & \uparrow & \\ & & & f^{-1}T_S & \xlongequal{\quad} & f^{-1}T_S & \\ & & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & \mathfrak{g}_f(F, \nabla_f) & \longrightarrow & \mathcal{E}_X^\sim(F, \nabla_f) & \longrightarrow & T_X^\sim \longrightarrow 0 \\ & & \parallel & & \uparrow \swarrow \nabla^\sim & & \uparrow \\ 0 & \longrightarrow & \mathfrak{g}_f(F, \nabla_f) & \longrightarrow & \mathcal{E}_f(F, \nabla_f) & \longrightarrow & T_f \longrightarrow 0 \\ & & \uparrow & & \uparrow \searrow \nabla_f & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

Now, we are ready to show the implication (1) \Rightarrow (2). The condition (1) guarantees the existence of a global $\nabla : F \rightarrow \mathcal{E}_X(F)$ inducing ∇_f . The restriction of ∇ to T_X^\sim gives a morphism of complexes $\nabla^\sim : T_X^\sim \rightarrow \mathcal{E}_X^\sim(F, \nabla_f)$ that lifts ∇_f in the diagram (4.1.7). Thus, the triple of maps $(\nabla_f, \nabla^\sim, \text{id}_{f^{-1}T_S})$ gives a morphism between the third and the second column of diagram (4.1.7) and a posteriori a morphism between the distinguished triangles in $D^b(S)$ which one obtains after applying f_* . In particular, this yields (2).

In order to prove that (2) implies (1) it is enough to show that the connection ∇_f lifts to a morphism ∇^\sim splitting the second row of diagram (4.1.7). Indeed, if ∇^\sim exists we can extend it by \mathcal{O}_X -linearity to a connection $\nabla : T_X \rightarrow \mathcal{E}_X(F)$. Furthermore ∇ ought to be integrable for ∇^\sim is a morphism of complexes.

Let

$$e_{(X/S;F,\nabla_f)} \in H^1(X, \text{Hom}_{f^{-1}\mathcal{O}_S}(f^{-1}T_S, \mathcal{E}_f(F, \nabla_f))) \text{ and } e_{X/S} \in H^1(X, \text{Hom}_{f^{-1}\mathcal{O}_S}(f^{-1}T_S, T_f))$$

denote the extension classes of the second and the third column of (4.1.7) respectively. The lifting ∇^\sim of ∇_f will exist if and only if the push-forward of the extension class $e_{X/S}$ via the morphism $\nabla_f : T_f \rightarrow \mathcal{E}_f(F, \nabla_f)$ coincides with $e_{(X/S;F,\nabla_f)}$.

Observe that the Lerray spectral sequence gives a diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ H^1(S, \Omega_S^1 \otimes f_*T_f) & \xrightarrow{h^1(f_*\nabla_f)} & H^1(S, \Omega_S^1 \otimes \mathbb{R}^0 f_*\mathcal{E}_f(F, \nabla_f)) \\ & \xleftarrow{h^1(f_*\sigma_f)} & \\ \downarrow & & \downarrow \\ H^1(X, \text{Hom}_{f^{-1}\mathcal{O}_S}(f^{-1}T_S, T_f)) & \xrightarrow{h^1(\nabla_f)} & H^1(X, \text{Hom}_{f^{-1}\mathcal{O}_S}(f^{-1}T_S, \mathcal{E}_f(F, \nabla_f))) \\ & \xleftarrow{h^1(\sigma_f)} & \\ \downarrow q & & \downarrow Q \\ H^0(S, \Omega_S^1 \otimes R^1 f_*T_f) & \xrightarrow{R^1 f_*\nabla_f} & H^0(S, \Omega_S^1 \otimes \mathbb{R}^1 f_*\mathcal{E}_f(F, \nabla_f)) \\ & \xleftarrow{R^1 f_*\sigma_f} & \end{array}$$

which is commutative for both σ_f and ∇_f . Set $e = h^1(\nabla_f)(e_{X/S}) - e_{(X/S;F,\nabla_f)}$. Since $q(e_{X/S}) = \kappa_{X/S}^1$ and $Q(e_{(X/S;F,\nabla_f)}) = \kappa_{(X/S;F,\nabla_f)}^1$ we have that

$$Q(e) = R^1 f_*\nabla_f \circ h^0(\kappa_{X/S}) - h^0(\kappa_{(X/S;F,\nabla_f)}) = 0$$

by condition (2). Thus $e \in H^1(S, \Omega_S^1 \otimes \mathbb{R}^0 f_*\mathcal{E}_f(F, \nabla_f))$. Due to the smoothness of the stack $\mathcal{M}_{\text{DR}}^o(X/S, n)$ we have an equality

$$\text{Ext}_{\mathcal{O}_S}^1(T_S, \mathbb{R}^0 f_*\mathcal{E}_f(F, \nabla_f)) = \text{Ext}_{\mathcal{O}_S}^1(\ker(\kappa_{(X/S;F,\nabla_f)}^1), \mathbb{R}^0 f_*\mathcal{E}_f(F, \nabla_f)).$$

On the other hand the condition (2) implies that the image of e in the group

$$\text{Ext}_{\mathcal{O}_S}^1(\ker(\kappa_{(X/S;F,\nabla_f)}^1), \mathbb{R}^0 f_*\mathcal{E}_f(F, \nabla_f))$$

is equal to zero which proves the proposition. \square

Remark 4.7 (i) The cohomology class e which appears in the proof of Proposition 4.6 is the obstruction for the relative connection (F, ∇_f) to lift to a connection on the whole X . To make this somewhat more explicit observe first that e lies even deeper inside $H^1(X, \text{Hom}_{f^{-1}\mathcal{O}_S}(f^{-1}T_S, \mathcal{E}_f(F, \nabla_f)))$. Indeed, diagram (4.1.7) implies that

$$e_{X/S} = h^1(\sigma_f)(e_{(X/S; F, \nabla_f)})$$

which combined with the identity $\sigma_f \circ \nabla_f = \text{id}$ gives $h^1(\sigma_f)(e) = 0$. Thus $e \in H^0(S, \Omega_S^1 \otimes \mathbb{R}^0 f_* \mathfrak{g}_f(F, \nabla_f))$. It is easy to represent e by a Čech cocycle. Let \mathfrak{U} be a Čech covering of S . Due to Proposition 4.6 for any $U \in \mathfrak{U}$ we can choose a lifting $\nabla_U^\sim : T_{X_U}^\sim \rightarrow \mathcal{E}_{X_U}^\sim(F, \nabla_f)$ of ∇_f . The difference $e_{UV} := \nabla_V^\sim - \nabla_U^\sim$ for two $U, V \in \mathfrak{U}$ is trivial on $T_f \subset T_X^\sim$ and hence belongs to $\Gamma(X_{U \cap V}, \text{Hom}_{f^{-1}\mathcal{O}_S}(f^{-1}T_S, \mathfrak{g}_f(F, \nabla_f))) = \Gamma(U \cap V, \Omega_S^1 \otimes \mathbb{R}^0 f_* \mathfrak{g}_f(F, \nabla_f))$. In particular, the cocycle $\{e_{UV}\} \in Z^1(\mathfrak{U}, \Omega_S^1 \otimes \mathbb{R}^0 f_* \mathfrak{g}_f(F, \nabla_f))$ represents e .

(ii) The proof of Proposition 4.6 explains also the geometric meaning of the natural weakening of condition (2) which takes into account only the naive parts of the Kodaira-Spencer maps. Namely we have the equivalence of the following two conditions

(1') *Locally in S there exists a global integrable connection $\nabla : T_X \rightarrow \mathcal{E}_X(F)$ which induces*

$$\nabla_f : T_f \rightarrow \mathcal{E}_f(F).$$

(2') *The following diagram of sheaves on S*

$$\begin{array}{ccc} T_S & \xrightarrow{\kappa_{(X/S; F, \nabla_f)}^1} & \mathbb{R}^1 f_* \mathcal{E}_f(F, \nabla_f) \\ & \searrow \kappa_{X/S}^1 & \nearrow R^1 f_* \nabla_f \\ & R^1 f_* T_f & \end{array}$$

commutes.

The implication (1') \Rightarrow (2') follows in exactly the same way as in the proof of Proposition 4.6. For the proof (2') \Rightarrow (1') one only has to notice that when S is replaced by an affine open $U \subset S$ and when X is replaced by $X \times_S U$ the obstruction class e vanishes since $H^1(U, \Omega_U \otimes \mathbb{R}^0 f_* \mathcal{E}_f(F, \nabla_f)) = 0$.

4.2 General variations - some speculations

In this section we probe a possible general framework for non-abelian Hodge theory in the "weight one" case. The section is not used in the rest of the paper and may be skipped by the reader.

The results of Simpson discussed in Section 3(NA(i-vi)) suggest the following general notion which is essentially due to Simpson:

Definition 4.8 *A polarized complex non-abelian Hodge structure of weight one consists of the following data*

CNAH1 (*Space*) A complex algebraic variety (or stack) M .

CNAH2 (*Hodge filtrations*) A variety Z equipped with:

- an algebraic \mathbb{C}^\times -action $\gamma : \mathbb{C}^\times \times Z \rightarrow Z$ such that for any closed point $z \in Z$ the limits $\lim_{t \rightarrow 0} \gamma_t(z)$ exists in Z ;
- a \mathbb{C}^\times -equivariant morphism $\zeta : Z \rightarrow \mathbb{P}^1$;
- an isomorphism $Z_1 \cong M$.

CNAH3 (*Opposedness of the Hodge filtrations*) There exists a real analytic trivialization $\phi : Z \rightarrow M \times \mathbb{P}^1$ such that

- The ϕ -constant sections of $\zeta : Z \rightarrow \mathbb{P}^1$ are holomorphic;
- If $D(\zeta)$ denotes the Douady space of sections of ζ and if $M^\phi \subset D(\zeta)$ is the subset of all ϕ -constant sections, then there exists a neighborhood $M^\phi \subset \mathcal{S} \subset D(\zeta)$ such that for every $x, y \in \mathbb{P}^1$ the natural evaluation map

$$\text{ev}_{x,y} : D(\zeta) \rightarrow Z_x \times Z_y$$

induces an isomorphism of \mathcal{S} and a neighborhood of the diagonal.

CNAH4 (*Polarization*) An algebraic relative form $\Omega \in H^0(Z, \Omega_\zeta^2 \otimes \zeta^* \mathcal{O}_{\mathbb{P}^1}(2))$ satisfying $\gamma_t^* \Omega = t\Omega$ for all $t \in \mathbb{C}^\times$ and such that the restriction $\Omega_t := \Omega|_{Z_t}$ is a symplectic form when restricted to the smooth locus of any fiber.

CNAH5 (*Splitting of the Hodge filtrations*) Two morphisms $h_0 : Z_0 \rightarrow B_0$ and $h_\infty : Z_\infty \rightarrow B_\infty$ whose fibers are Lagrangian for Ω_0 and Ω_∞ respectively and generically transversal to the closures of the \mathbb{C}^\times -orbits.

A polarized complex non-abelian Hodge structure will be called *real* if in addition Z is equipped with an antiholomorphic involution $\sigma : Z \rightarrow Z$ covering the antipodal involution on \mathbb{P}^1 so that ζ and Ω are real and $\mathcal{S}^\sigma = M^\phi$.

Remark 4.9 (i) In fact Simpson [Simpson97a, Section 5] defines a filtration of a scheme (or a stack) M as a \mathbb{G}_m -scheme (or stack) Z equipped with a \mathbb{G}_m -equivariant map to \mathbb{A}^1 and an identification $Z_1 \cong M$. In view of this condition **CNAH2** can be thought of as giving two filtrations on M - one corresponding to $Z_{|\mathbb{P}^1 \setminus \{0\}}$ and the other to $Z_{|\mathbb{P}^1 \setminus \{\infty\}}$

(ii) The notion of an abstract non-abelian Hodge structure of weight one proposed above specializes to some well known geometric objects. For example if M is smooth, then a real non-abelian Hodge structure of weight one is nothing but the but a hyperkähler structure with \mathbb{C}^\times -equivariant twistor space⁴ This follows simply from the twistor interpretation of

⁴This shouldn't be confused with the much stronger notion of a hyperkähler cone used say in [Brylinski98b].

hyperkähler structures from [Hitchin *et al.*87]. In fact in this case one can replace condition **CNAH3** by the much simpler to check (but equivalent condition) that ζ has at least one σ -invariant holomorphic section with normal bundle isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^{\dim Z/\mathbb{P}^1}$.

Similarly if we drop conditions **CNAH4** and **CNAH5** and the \mathbb{C}^\times -equivariance condition from **CNAH2** then we recover the notion of a (possibly singular) quaternionic variety introduced by Deligne in [Deligne89].

(iii) The twistor family $\zeta : Z \rightarrow \mathbb{P}^1$ corresponding to a non-abelian Hodge structure has many peculiar geometric properties.

For example the assumption that Z is algebraic immediately implies that Z and a posteriori M cannot be proper. In addition the \mathbb{C}^\times -equivariance required in **CNAH2** implies that ζ trivializes holomorphically over $\mathbb{P}^1 \setminus \{0, \infty\}$. In particular $M = Z_1, Z_0$ and Z_∞ are the only non-isomorphic members of the family ζ .

If in addition we assume that M is smooth then the condition that for every $z \in Z$ the limit $\lim_{t \rightarrow 0} tz$ exists, then by [Kaledin97, Section 4] we see that Z_0 is birational to the total space T of the cotangent bundle of $Z_0^{\mathbb{C}^\times}$ and that moreover the symplectic form Ω_0 transforms into the standard symplectic form on T at least over a big open set. In particular the universal categorical quotient of Z_0 by \mathbb{C}^\times exists and is birational to $Z_0^{\mathbb{C}^\times}$. By the \mathbb{C}^\times -equivariance of ζ then it is clear that the universal categorical quotient of $Z_{|\mathbb{P}^1 \setminus \{0\}}$ by \mathbb{C}^\times will exist and will be birational to $Z_0^{\mathbb{C}^\times}$ as well. Thus we get a rational map $Z_{|\mathbb{P}^1 \setminus \{0\}} \rightarrow Z_0^{\mathbb{C}^\times} \times \mathbb{P}^1$ which furnishes the family of algebraic symplectic family $Z_{|\mathbb{P}^1 \setminus \{0\}} \rightarrow \mathbb{P}^1$ with a Lagrangian foliation which on the fiber Z_0 is generically transversal to h_0 . Similar analysis holds over $\mathbb{P}^1 \setminus \{\infty\}$.

This structure of Z is consistent with the picture we got in Section 3.1(**NA(iv)**).

Definition 4.8 seems (and is in fact) quite restrictive. Some obvious examples of non-abelian Hodge structures of weight one are:

Example 4.10 Every polarized complex hodge structure of weight one (= a regular equivariant quaternionic vector space [Simpson97b] [Kaledin97, Section 1.1.7]) is also a non-abelian polarized Hodge structure of weight one.

Example 4.11 Let V be an $n + 1$ -dimensional real vector space and let $V_{\mathbb{C}} : V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Put $P := \text{Proj}(S^\bullet V_{\mathbb{C}}^\vee)$ for the projectivization of $V_{\mathbb{C}}$. Take $M \subset P \times P^\vee$ to be the affine subvariety defined as $M = \{(x, h) | x \notin h\}$. The Euler sequence on P identifies M with the twisted cotangent bundle of P corresponding to the hyperplane class $c_1(\mathcal{O}_P(1)) \in H^1(P, \Omega_P^1)$. In particular there is a tautological holomorphic family $Z^+ \rightarrow \mathbb{A}^1 = H^1(P, \Omega_P^1)$ of T_P^\vee -torsors on P . By construction we have

$$Z_{|\mathbb{A}^1 \setminus \{0\}}^+ \cong \mathbb{A}^1 \setminus \{0\} \times M \text{ and } Z_0^+ = T_P^\vee.$$

Moreover we can use the real structure on $V_{\mathbb{C}}$ and the antipodal real structure on \mathbb{P}^1 to glue Z^+ with its conjugate family Z^- and obtain a \mathbb{C}^\times -equivariant family $\zeta : Z \rightarrow \mathbb{P}^1$. The

family ζ is the twistor family of the hyperkähler manifold T_P^\vee which is obtained from the quaternionic space \mathbb{H}^n by hyperkähler reduction by $U(n)$ [Calabi79]. In particular **CNAH1-4** will automatically hold. To see that **CNAH5** holds observe that P is a toric variety and so we have a Poisson action of $(\mathbb{C}^\times)^n$ on T_P^\vee for the standard algebraic symplectic form. Define $h_0 : T_P^\vee \rightarrow \mathbb{C}^n$ to be just the moment map for the $(\mathbb{C}^\times)^n$ -action. It is well known that the fibers of h_0 are coisotropic (see e.g. [Chriss-Ginzburg97, Section 1.5]) and it is not hard to see that the fiber of h_0 over $0 \in \mathbb{C}^n$ is a (reducible) subvariety of dimension n . Thus the generic fiber of h_0 is Lagrangian and generically transversal to the fibers of $T_P^\vee \rightarrow P$.

Example 4.12 Similarly to Example 4.11 one can start not with a complex projective space but with a principally polarized abelian variety (A, θ) . In this case one takes M to be the twisted cotangent bundle of A corresponding to the class $c_1(\theta) \in H^1(A, \Omega_A^1)$ and $Z^+ \rightarrow \mathbb{C}\theta \subset H^1(A, \Omega_A^1)$ to be again the universal torsor. Again Z^+ glues with its complex conjugate to produce the twistor space of the hyperkähler manifold T_A^\vee . The hyperkähler structure on T_A^\vee exists since the Kähler metric corresponding to θ is flat (see [Calabi79] or [Biquard-Gauduchon97]). Finally the splitting of the Hodge filtrations follows from the fact that T_A^\vee is a trivial bundle and so we can take h_0 to be the projection onto the fiber $T_{A,0}^\vee$ at 0.

Example 4.13 Given a smooth projective variety X/\mathbb{C} and a complex reductive algebraic group G , then we get a non-abelian Hodge structure by taking $M = H_{\text{DR}}^1(X, G)$, $Z = H_{\text{Del}}^1(X, G)$ and the twistor lines come from the choice of harmonic metric (these are called the preferred sections in [Simpson97a]). As explained in detail in [Simpson97a] (see also Section 3.1(**NA(vi)**)) the fibers of $H_{\text{Del}}^1(X, G)$ over 0 and ∞ are naturally identified with $H_{\text{Dol}}^1(X, G)$ and $\overline{H}_{\text{Dol}}^1(X, G)$ respectively and the splitting of the Hodge filtrations is given by the Hitchin map at 0 and its complex conjugate at ∞ .

Note that in fact all of the previous examples are to some extent special cases of Example 4.13. Indeed Examples 4.10 and 4.12 correspond to e.g. taking X to be an abelian variety of dimension g (or a smooth curve of genus g) and $G = \mathbb{C}$ or $G = \mathbb{C}^\times$ respectively. For Example 4.11 a slightly different approach is necessary. Fix an elliptic curve E and let $G = \text{SL}_n(\mathbb{C})$. In this case the moduli space $H^1(E, \text{SL}_{n+1}(\mathcal{O}_E))$ of semistable vector bundles of rank $n+1$ and degree 0 can be naturally identified (see e.g. [Tu93]) with the symmetric product $E^{(n)}$ through the Fourier-Mukai transform on E . On the other hand one can use the Abel-Jacobi map for E to identify $E^{(n)}$ with an n -dimensional complex projective space P and so we have an S_n -Galois cover $\pi : E^{\times n} \rightarrow P$, where S_n denotes the symmetric group on n letters. It is clear that the moduli space $H_{\text{Dol}}^1(E, \text{SL}_{n+1}(\mathbb{C}))$ of Higgs bundles is birational⁵

⁵In fact for $n = 1$ the moduli space $H_{\text{Dol}}^1(E, \text{SL}_2(\mathbb{C}))$ can be naturally identified with the Hilbert quotient of $T^\vee E$ by $\{\pm 1\}$ i.e. with the blow up of the ordinary quotient $T^\vee E / \{\pm 1\}$ at its four ordinary double points. It is reasonable to expect therefore that more generally $H_{\text{Dol}}^1(E, \text{SL}_{n+1}(\mathbb{C}))$ will be just the Hilbert quotient of $T^\vee E^{\times n}$ by S_n but this is irrelevant for our discussion.

to the S_n -quotient of the cotangent bundle to the abelian variety $E^{\times n}$ and hence birational to T_P^\vee . This gives a hyperkähler identification of big open sets of T_P^\vee and $H_{\text{Dol}}^1(E, \text{SL}_{n+1}(\mathbb{C}))$. The corresponding splittings of the Hodge filtrations seem to be different however. Indeed the Lagrangian fibration for $H_{\text{Dol}}^1(E, \text{SL}_{n+1}(\mathbb{C}))$ is given by the Hitchin map and hence has fibers abelian varieties. In contrast the Lagrangian fibration for T_P^\vee has toric varieties as fibers.

In view of the above discussion it seems reasonable to try to relax the conditions **CNAH1-5**. For example instead of working with a \mathbb{C}^\times -scheme (or stack) M one may work with a formal scheme (or a formal stack) equipped with a \mathbb{C}^\times action. Similarly it seems reasonable to only require the existence of rational maps $h_0 : Z_0 \dashrightarrow B_0$ and $h_\infty : Z_\infty \dashrightarrow B_\infty$ with Lagrangian fibers. With these relaxations one can now speculate about the existence of more exotic examples.

Example-Speculation 4.14 Let G be a complex semi simple group and let $P \subset G$ be a parabolic. It is known [Biquard-Gauduchon97, Theorem 1] (see also [Kaledin97] and [Feix99]) that for any homogeneous Kähler metric g on the partial flag variety G/P there exists a unique G -invariant hyperkähler metric on $T_{G/P}^\vee$ which restricts to g on the zero section. As shown in [Biquard98, Section 3.3] the twistor space Z of such a hyperkähler metric again has the property that $Z_{|\mathbb{P}^1 \setminus \{\infty\}}$ is the tautological family of $\Omega_{G/P}^1$ -torsors with class $[\omega] \in H^1(G/P, \Omega_{G/P}^1)$ where ω is the Kähler class of g . In particular we can take M to be the twisted cotangent bundle $T_{G/P}^\vee([\omega])$. In fact this description can be used [Mirkovic96] as a starting point for a completely algebraic description of the hyperkähler structure on $T_{G/P}^\vee$ which is especially well suited for inducing hyperkähler structures on associated spaces.

To interpret this data as a non-abelian Hodge structure of weight one we also need to specify the splitting of the Hodge filtrations. For this we may proceed in two ways. One alternative is to replace $Z_0 = T_{G/P}^\vee$ with a formal neighborhood of the zero section in $T_{G/P}^\vee$. In that case M should be replaced by the scheme parameterizing all pairs (x, γ) where $x \in G/P$ and γ is a section of $T_{G/P}^\vee([\omega]) \rightarrow G/P$ over a formal neighborhood of $x \in G/P$ (compare with Remark 2.9(ii)). In this case the splitting of the Hodge filtration on say Z_0 will be just the restriction of the moment map $\mu : T_{G/P}^\vee \rightarrow \text{Lie}(G)^\vee$ for the standard G -action and symplectic form on $T_{G/P}^\vee$. Another possibility is to keep Z as it is and to look for a Lagrangian foliation transversal to the fibers of the cotangent bundle. One choice might come from taking preimages of P -coadjoint orbits via the moment map $\mu_P : T_{G/P}^\vee \rightarrow \text{Lie}(P)^\vee$.

Example-Speculation 4.15 A variation of the previous example is to take Z to be the twistor space of the Kronheimer hyperkähler metric [Kronheimer90], [Biquard98] of a coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^\vee$ of a complex reductive Lie algebra \mathfrak{g} . In this case the twistor fiber Z_1 is naturally identified with the moduli space of solutions of the twisted complex Nahm equations or equivalently with the moduli space of rotationally symmetric logarithmic λ connections on \mathbb{C}^\times [Biquard98].

The latter interpretation suggests that it may be possible to define the splitting map $h_0 : \mathbb{O} \rightarrow B_0$ by working with Jacobians of spectral covers of logarithmic spaces. This is a very interesting question which merits serious consideration.

Example-Speculation 4.16 Let C be a smooth curve of genus g and let $Y = \text{tot}(T_C^\vee)$ be the total space of its cotangent bundle. The surface Y is naturally holomorphic symplectic and so the Hilbert scheme $Y^{[m]}$ of 0-dimensional schemes of length m on Y will be a smooth holomorphic symplectic manifold. In fact due to [Kaledin97, Theorem 1] one knows that Y carries an incomplete $U(1)$ -equivariant hyperkähler metric and so one expects that $Y^{[m]}$ is hyperkähler as well. The analysis of [Kaledin97] and the twistor space construction in the forthcoming thesis [Feix99] show that at least when one works with the formal completion of the zero section in Y the corresponding twistor family behaves exactly as required by **CNAH1-4**. More precisely given any smooth Kähler manifold (V, ω) whose Kähler metric is real analytic [Kaledin97, Theorem 1] and [Feix99] construct a unique $U(1)$ -equivariant hyperkähler structure on a tubular neighborhood of the zero section of T_V^\vee which restricts to ω . Furthermore it is shown in [Feix99] (and implicitly in [Kaledin97]) that the twistor space of this hyperkähler structure is trivial over \mathbb{C}^\times with a fiber which is biholomorphic to a tubular neighborhood N of V in the ω -twisted cotangent bundle $T_{\omega, V}^\vee$. Here $V \subset \text{tot}(T_{\omega, V}^\vee)$ is embedded as a real-analytic submanifold via the section corresponding to the Kähler form ω . Moreover Kaledin's interpretation of the points of N as regular extended connections on V shows that N can be identified with a tubular neighborhood of the diagonal of $V \times V$ and so the twistor space of N at infinity can be interpreted as the complex conjugate of the deformation of $V \subset V \times V$ to its normal cone. In these setup the twistor lines for N become just the intrinsic holomorphic exponential map for the Kähler manifold V as described by [Kapranov99, Section 2.9] (see also [Bershadsky *et al.*94], [Calabi79]).

One might also speculate that there should be a natural Lagrangian foliation of $Y^{[m]}$ which is transversal to the fibers of the Lagrangian fibration $Y^{[m]}$ and can serve as a splitting of the Hodge filtrations (at least in the formal case).

This is supported further by the beautiful analysis of [Hurtubise96] (see also [Nakajima96, Chapter 7]) which shows that for infinitely many choices of m the Hilbert scheme $Y^{[m]}$ is birational to the moduli space $H_{\text{Dol}}^1(C, \text{GL}_n(\mathbb{C}))$ and that the holomorphic symplectic forms match on a big open set after this birational identification.

Following [Simpson97a, Section 8] one can speak of variations of non-abelian Hodge structures of weight one. By definition this is a morphism of varieties $\zeta : Z \rightarrow S \rightarrow \mathbb{P}^1$, so that

- Z is equipped with a \mathbb{C}^\times -action covering the standard action on \mathbb{P}^1 and such that for every $s \in S$ the restriction $Z_{|_{\{s\} \times \mathbb{P}^1}}$ satisfies **CNAH1-4**.
- There exist schemes $B_0 \rightarrow S$ and $B_\infty \rightarrow S$ and morphisms $h_0 : Z_{|_{S \times \{0\}}} \rightarrow B_0$ and $h_\infty : Z_{|_{S \times \{\infty\}}} \rightarrow B_\infty$ over S specializing to splittings of the Hodge filtrations for every $s \in S$.

- The family $Z_{|S \times \{1\}} \rightarrow S$ is a local system of schemes on S , i.e. $Z_{|S \times \{1\}}$ is equipped with an action of the formal groupoid S_{DR} over S 2.12.
- Griffiths transversality holds, i.e. the natural action of the formal groupoid S_{DR} on $Z_{|S \times \{1\}}$ extends to an action of the Hodge formal groupoid S_{Hod} on $Z_{|S \times (\mathbb{P}^1 \setminus \infty)}$. Here $S_{\text{Hod}} := N \rightrightarrows S \times \mathbb{A}^1$ with N being the formal completion of S inside its deformation to the normal cone.

Moreover given a non-abelian Hodge structure $(M, Z \rightarrow \mathbb{P}^1, \phi, \Omega, h_0, h_\infty)$ of weight one we can talk of non-abelian (p, p) classes in exactly the same way as before. Namely these are the closed points in M that correspond to \mathbb{C}^\times fixed points in Z_0 under the trivialization ϕ .

In particular it makes sense to ask if Theorem A will hold in this more general context. We formulate this as a question for further study:

Question 4.17 *Let $Z \rightarrow S \times \mathbb{P}^1$ be a variation of non-abelian Hodge structures of weight one over a smooth quasi projective S . Assume that we are given section a of $Z_{|S \times \{1\}} \rightarrow S$ which is horizontal for the non-abelian connection, i.e. which comes from a section of $Z_{|S \times \{1\}} \rightarrow S_{\text{DR}}$. Assume further that for some point $o \in S$ the point $a(o)$ is a non-abelian (p, p) class in $Z_{(o, 1)}$ is it true that $a(s)$ is a non-abelian (p, p) -class for all $s \in S$?*

It will be very interesting to investigate this question for some example which is not of geometric origin. For this one will need to build interesting variations of non-abelian Hodge structures. For almost all examples considered above this can be done in the obvious manner. In that respect Example-Speculation 4.16 looks especially promising. Indeed one expects that for any smooth family of curves $C \rightarrow S$ and any global Kähler metric on C the corresponding family of hyperkähler structures produced by Kaledin's construction will constitute a variation of non-abelian Hodge structures of weight one. Question 4.17 becomes very intriguing in this setup since Nakajima had shown [Nakajima96, Proposition 7.5] that

$$(Y^{[n]})^{\mathbb{C}^\times} = \coprod_{\nu} C^{(\nu)}$$

where ν runs over all partitions of n and

$$C^{(\nu)} := C^{(\alpha_1)} \times \dots \times C^{(\alpha_n)} \text{ for a partition } \nu = (1^{\alpha_1}, 2^{\alpha_2}, \dots, n^{\alpha_n}).$$

Similarly one might attempt to construct examples of non-abelian variations starting from Example-Speculation 4.15. One possibility is to work with a local system of complex reductive Lie algebras and to consider the minimal nilpotent orbit in each fiber. Since the Lie bracket is assumed to be horizontal for the corresponding connection it is clear then that the connection will preserve the minimal nilpotent orbit. Thus the family of minimal nilpotent orbits will give us a local system of schemes. Moreover the analysis carried out in [Brylinski98b] [Brylinski98a] shows that in several cases the zero fiber of the twistor space is isomorphic (at least locally in the étale topology) to the total space of the

holomorphic cotangent bundle of the real minimal nilpotent orbit equipped with its natural Kähler structure. Another possibility suggested from the geometric quantization approach [Brylinski98b] is to work with a fixed complex nilpotent orbit but to vary the holomorphic polarization on it. Then from the geometric quantization point of view one expects a flat connection on the space of bundle of quantization spaces over the parameter space of all polarizations. Since the quantization spaces are typically spaces of functions (or sections in a line bundle) on the variety it is not unreasonable to look for a connection on the actual family of polarized orbits. One such connection with finite monodromy is explicitly described in [Brylinski97].

5 The main theorem

As explained in Section 3.1(NA(v)) Simpson characterizes [Simpson92] the non-abelian (p, p) classes as local systems whose Higgs bundles are fixed under the natural \mathbb{C}^\times -action. Thus Theorem A can be viewed as a statement comparing the non-abelian Gauss-Manin connection on the stack $\mathcal{M}_{\text{DR}}(X/S, n)$ of relative local systems with the \mathbb{C}^\times -action of on the stack $\mathcal{M}_{\text{Dol}}(X/S, n)$ of relative Higgs bundles. It is hard to work out such a comparison in practice because the homeomorphism τ_X in the non-abelian Hodge theorem (see 3.1(NA(iii))) is incompatible with both the Gauss-Manin connection and the \mathbb{C}^\times -action.

To circumvent such difficulties we use (see Section 5.1) the fact that the \mathbb{C}^\times -fixed points can be detected infinitesimally. This reduces the question to a cohomological calculation for ordinary local systems of vector spaces which can then be carried out by using Simpson's higher Kähler identities [Simpson92, Lemma 2.2].

It is instructive to first analyze the case when we are dealing with sections that pass through smooth points of the moduli stack. This is the subject of Section 5.2. Finally in Section 5.3 we complete the proof of Theorem A by reducing the general case to the smooth situation.

5.1 The Euler vector field

Let M be a scheme over $\text{Spec}(\mathbb{C})$ which is equipped with an algebraic \mathbb{C}^\times -action $\mu : \mathbb{C}^\times \times M \rightarrow M$. Since the group $\mathbb{C}^\times = \text{Spec}(\mathbb{C}[t, t^{-1}])$ is parallelizable the action μ induces a canonical Euler vector field $\eta \in H^0(M, T_M)$ defined as $\eta := i^*(d\mu)(d/dt, 0)$, with $i : M \hookrightarrow \mathbb{C}^\times \times M$, $i(x) := (1, x)$ being the natural inclusion. From the definition of η it is clear that a smooth closed point $x \in M$ is \mathbb{C}^\times -fixed if and only if the image η_x of η under the natural evaluation map $\text{ev}_x : T_M \rightarrow T_{M, x} := T_M \otimes k(x)$ is zero.

More generally consider a smooth Artin stack \mathcal{X} . Assume further that \mathcal{X} is equipped with a \mathbb{C}^\times -action $\mu : \mathbb{C}^\times \times \mathcal{X} \rightarrow \mathcal{X}$. Then we can again use the flow to construct the Euler vector field for μ which in this case will be a 1-morphism of stacks $\eta : \mathcal{X} \rightarrow T_{\mathcal{X}}$ which is a section of the structure morphism $\text{pr}_{\mathcal{X}} : T_{\mathcal{X}} \rightarrow \mathcal{X}$. The \mathbb{C}^\times -fixed sub stack $\mathcal{X}^{\mathbb{C}^\times}$ of \mathcal{X} is defined as the stacky zero locus of the vector field η . More precisely $\mathcal{X}^{\mathbb{C}^\times} := \mathcal{X} \times_{\text{ver}_{\mathcal{X}}, T_{\mathcal{X}}, \eta} \mathcal{X}$ where as usual $\text{ver}_{\mathcal{X}}$ is the vertex of the cone stack $T_{\mathcal{X}}$ (see A.2 for notation).

Remark 5.1 Note that if \mathcal{X} is of the form $[R/G]$ with R a quasi projective scheme over \mathbb{C} and G a complex reductive group and if in addition the \mathbb{C}^\times -action on \mathcal{X} descends from an action of $G \times \mathbb{C}^\times$ on R , then the sub stack $\mathcal{X}^{\mathbb{C}^\times}$ is just the quotient stack $[R^{\mathbb{C}^\times}/G]$.

Let now S be a smooth complex quasi-projective variety and let $f : X \rightarrow S$ be a smooth projective morphism. Consider the stack $\pi_{\text{DR}} : \mathcal{M}_{\text{DR}}(X/S, n) \rightarrow S$ of relative rank n local systems and the stack $\pi_{\text{Hod}} : \mathcal{M}_{\text{Hod}}(X/S, n) \rightarrow S \times \mathbb{A}^1$ of λ -connections of rank n (see [Simpson97a, Proposition 4.1] for existence). Let $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}^o(X/S, n)$ be an algebraic section which is horizontal with respect to the non-abelian Gauss-Manin connection.

For future reference we formulate the following lemma which is well known to the experts:

Lemma 5.2 *Assume S is projective.*

- (i) *There exists a canonical extension of a_{DR} to a section $a_{\text{Hod}} : S \times \mathbb{A}^1 \rightarrow \mathcal{M}_{\text{Hod}}^o(X/S, n)$. In particular there is an algebraic section $a_{\text{Dol}} : S \rightarrow \mathcal{M}_{\text{Dol}}^o(X/S, n)$ which corresponds to a_{DR} through the non-abelian Hodge theorem.*
- (ii) *For a point $s \in S$ the local system $a_{\text{DR}}(s) \in \mathcal{M}_{\text{DR}}(X_s, n)$ will underly a \mathbb{C} VHS iff $a_{\text{Dol}}(s) \in \mathcal{M}_{\text{Dol}}(X_s, n)^{\mathbb{C}^\times}$, i.e. iff the Euler vector field η vanishes at $a_{\text{Dol}}(s)$.*

Proof. Let (F, ∇_f) be a bundle with a relative integrable connection on X representing the section a_{DR} . The horizontality of a_{DR} with respect to the non-abelian Gauss-Manin connection implies (see Proposition 4.6(1)) the existence of a global integrable connection $\nabla : F \rightarrow F \otimes \Omega_X^1$ which induces ∇_f . Since S (and therefore X) is projective we can invoke [Simpson92, Corollary 3.10] and conclude that (F, ∇) will correspond to a global Higgs bundle (E, θ) on X which is filtered by Higgs subbundles and for which the associated graded is a direct sum of stable Higgs bundles with vanishing rational Chern classes. Using (F, ∇) and (E, θ) one can produce a twistor line $\mathbb{A}^1 \rightarrow \mathcal{M}_{\text{Hod}}(X, n)$ in exactly the same way as in [Simpson97a, Section 4]. Namely if $\bar{\partial}_F$ and $\bar{\partial}_E$ denote the complex structure operators for F and E respectively we define $\nabla_\lambda := \lambda \nabla + (1 - \lambda)\theta$ and $\bar{\partial}_\lambda := \lambda \bar{\partial}_F + (1 - \lambda)\bar{\partial}_E$. The section $\mathbb{A}^1 \rightarrow \mathcal{M}_{\text{Hod}}(X, n)$, $\lambda \mapsto (\bar{\partial}_\lambda, \nabla_\lambda)$ gives then the twistor line in question. Consider the restriction $a : S \rightarrow \mathcal{M}_{\text{Hod}}(X/S, n)$ of the family of global λ -connections $\bar{\partial}_\lambda + \nabla_\lambda$ to a family of relative λ -connections along the fibers of f . Therefore part (i) of the lemma will be proven if we can show that a will factor as

$$S \xrightarrow{a_{\text{Hod}}} \mathcal{M}_{\text{Hod}}^o(X/S, n) \hookrightarrow \mathcal{M}_{\text{Hod}}(X/S, n)$$

where as usual $\mathcal{M}_{\text{Hod}}^o(X/S, n) \subset \mathcal{M}_{\text{Hod}}(X/S, n)$ denotes the part of $\mathcal{M}_{\text{Hod}}(X/S, n)$ where π_{Hod} is smooth. This follows from the important fact (proven in the subsection "Griffiths transversality revisited" of [Simpson97a, Section 11]) that the map $\pi_{\text{DR}} : \mathcal{M}_{\text{Hod}}^o(X/S, n) \rightarrow S$ is trivial étale locally on $\mathcal{M}_{\text{Hod}}^o(X/S, n)$. Alternatively one can use the fact that a comes from a section $\mathbb{A}^1 \rightarrow \mathcal{M}_{\text{Hod}}(X, n)$ and [Simpson97a, Theorem 9.1].

The statement in part (ii) is just [Simpson92, Lemma 4.1] combined with Remark 5.1 and [Simpson92, Lemma 3.5].

□

The previous lemma shows that in order to understand the relation between the non-abelian (p, p) classes and the non-abelian Gauss-Manin connection we have to study the way η interacts with the non-abelian Gauss-Manin connection. For this we will need a description of η similar to the concrete description of $T_{\pi_{\text{Dol}}}$ obtained in Lemma 4.2.

Let $\mathcal{B}un(X/S, n)$ denote the moduli stack of rank n vector bundles on X with vanishing rational Chern classes along the fibers of f . Let $\pi : \mathcal{B}un(X/S, n) \rightarrow S$ be the structure map and let $\mathcal{B}un^o(X/S, n)$ denote the open sub stack in $\mathcal{B}un(X/S, n)$ over which the morphism π is smooth. The forgetting the Higgs field induces a morphism of stacks $p : \mathcal{M}_{\text{Dol}}(X/S, n) \rightarrow \mathcal{B}un(X/S, n)$ which, as explained in Section 3.1(**NA(iv)**) should be thought of as the projection to the $(1, 0)$ part of the non-abelian Dolbeault cohomology. By definition the action of \mathbb{C}^\times on $\mathcal{M}_{\text{Dol}}(X/S, n)$ respects p and so η can be interpreted as a section of the relative tangent stack $T_p := T_{\mathcal{M}_{\text{Dol}}(X/S, n)/\mathcal{B}un(X/S, n)} \rightarrow \mathcal{M}_{\text{Dol}}(X/S, n)$.

According to Lemma 4.2(ii) we have

$$T_{\pi_{\text{Dol}}} = h^1/h^0(\tau_{\leq 1} \mathbb{R}f_{\text{Dol}*} \mathfrak{g}_{f_{\text{Dol}}}(E_{\text{un}}, \theta_{\text{un}})),$$

and so given an S -point $a_{\text{Dol}} : S \rightarrow \mathcal{M}_{\text{Dol}}^o(X/S, n)$ represented by a relative Higgs bundle $(E, \theta_f) \rightarrow X$ we have $a_{\text{Dol}}^* T_{\pi_{\text{Dol}}} = h^1/h^0(\tau_{\leq 1} \mathbb{R}f_* \mathfrak{g}_f(E, \theta_f))$. The explicit description of η we need is now given by the following

Lemma 5.3 *Let (E, θ_f) be a Higgs bundle on X/S representing a section $a_{\text{Dol}} : S \rightarrow \mathcal{M}_{\text{Dol}}^o(X/S, n)$. Then*

(i) *There is an isomorphism of Picard stacks over S*

$$a_{\text{Dol}}^* T_p = h^1/h^0(\tau_{\leq 1} \text{Cone}[\mathbb{R}f_* \sigma_{\leq 0} \mathfrak{g}_f(E, \theta_f)[-1] \rightarrow \mathbb{R}f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f)]),$$

where σ denotes the stupid truncation.

(ii) *The section $a_{\text{Dol}}^* \eta : S \rightarrow a_{\text{Dol}}^* T_p \hookrightarrow a_{\text{Dol}}^* T_{\pi_{\text{Dol}}}$ corresponds to the natural morphism of complexes on X :*

$$\mathcal{O}_X[-1] \xrightarrow{\theta_f} \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f)$$

Proof. By definition E_{un} is the pull-back from the universal bundle on $\mathcal{B}un(X/S, n) \times_S X$ via the projection $p \times \text{id}_X : \mathcal{M}_{\text{Dol}}(X/S, n) \times_S X \rightarrow \mathcal{B}un(X/S, n) \times_S X$. Furthermore the Čech calculation in the proof of Lemma 4.2 shows that the pullback of the differential morphism

$dp : T_{\pi_{\text{Dol}}} \rightarrow p^*T_\pi$ via a_{Dol}^* is induced from the natural morphism of complexes

$$(5.1.8) \quad \mathfrak{g}_f(E, \theta_f) = \left[\begin{array}{c} \text{End}(E) \\ \text{ad}_{\theta_f} \downarrow \\ \text{End}(E) \otimes \Omega_f^1 \\ \text{ad}_{\theta_f} \downarrow \\ \text{End}(E) \otimes \Omega_f^2 \\ \text{ad}_{\theta_f} \downarrow \\ \vdots \end{array} \right] \longrightarrow \left[\begin{array}{c} \text{End}(E) \\ \downarrow \\ 0 \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \end{array} \right] = \sigma_{\leq 0} \mathfrak{g}_f(E, \theta_f)$$

On the other hand the morphism (5.1.8) fits in a short exact sequence of complexes

$$0 \rightarrow \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) \rightarrow \mathfrak{g}_f(E, \theta_f) \rightarrow \sigma_{\leq 0} \mathfrak{g}_f(E, \theta_f) \rightarrow 0$$

which pushes forward to a distinguished triangle in $D^b(S)$:

$$(5.1.9) \quad \mathbb{R}f_* \sigma_{\leq 0} \mathfrak{g}_f(E, \theta_f)[-1] \rightarrow \mathbb{R}f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) \rightarrow \mathbb{R}f_* \mathfrak{g}_f(E, \theta_f) \rightarrow \mathbb{R}f_* \sigma_{\leq 0} \mathfrak{g}_f(E, \theta_f).$$

The pullback of T_p via the section a_{Dol} is naturally identified with the fiber product

$$a_{\text{Dol}}^* T_{\pi_{\text{Dol}}} \times_{a_{\text{Dol}}^*(dp), (p \circ a_{\text{Dol}})^* T_\pi, \text{ver}_{(p \circ a_{\text{Dol}})^* T_\pi}} S,$$

which due to (5.1.9) and the functoriality of h^1/h^0 -construction coincides with

$$h^1/h^0(\tau_{\leq 1} \text{Cone}[\mathbb{R}f_* \sigma_{\leq 0} \mathfrak{g}_f(E, \theta_f)[-1] \rightarrow \mathbb{R}f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f)]).$$

This proves part (i) of the lemma.

The proof of part (ii) is almost tautological. Indeed the group \mathbb{C}^\times acts on $\mathcal{M}_{\text{Dol}}(X/S, n)$ by rescaling the Higgs fields and so the flow vector field $a_{\text{Dol}}^* \eta$ corresponds to the global section $\theta_f \in H^0(X, \text{End}(E) \otimes \Omega_f^1)$ which is naturally interpreted as a morphism $\theta_f : \mathcal{O}_X[-1] \rightarrow \sigma_{\geq 1} \mathfrak{g}_{X/S}(E, \theta_f)$. Therefore in terms of the cohomological description (i) of $a_{\text{Dol}}^* T_p$ the section $a_{\text{Dol}}^* \eta : S \rightarrow a_{\text{Dol}}^* T_p$ corresponds to the $\tau_{\leq 1}$ truncation of the morphism

$$Rf_* \mathcal{O}_X[-1] \rightarrow \mathbb{R}f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) \rightarrow \text{Cone}[\mathbb{R}f_* \sigma_{\leq 0} \mathfrak{g}_f(E, \theta_f)[-1] \rightarrow \mathbb{R}f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f)].$$

This finishes the proof of the lemma. \square

The $\tau_{\leq 1}$ truncation of the push forward of the morphism $\theta_f : \mathcal{O}_X[-1] \rightarrow \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f)$ induces naturally a global section $R^1 f_* \theta_f$ of the coherent sheaf $\mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f)$ on S . Let

$$\epsilon_f \in H^0(S, \mathbb{R}^1 f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) / f_* \text{End}(E))$$

denote the image of $R^1 f_* \theta_f$. The previous lemma has the following simple but important corollary.

Corollary 5.4 *Let $f : X \rightarrow S$ be a smooth projective family over a smooth quasi-projective scheme S . Let (E, θ_f) be a relative Higgs bundle on X representing a section $a_{\text{Dol}} : S \rightarrow \mathcal{M}_{\text{Dol}}^0(X/S, n)$. Then for point $s \in S$ we have $a_{\text{Dol}}(s) \in \mathcal{M}_{\text{Dol}}(X/S, n)^{\mathbb{C}^\times}$ iff $\epsilon_f(s) = 0$.*

Proof. By construction $\mathcal{H}^0 \mathbb{R}f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) = 0$ and hence we have a quasi-isomorphism

$$\begin{aligned} \tau_{\leq 1} \text{Cone}[\mathbb{R}f_* \sigma_{\leq 0} \mathfrak{g}_f(E, \theta_f)[-1] \rightarrow \mathbb{R}f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f)] &\cong (\mathbb{R}^1 f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) / \mathbb{R}^0 f_* \mathfrak{g}_f(E, \theta_f))[-1] \\ &= (\mathbb{R}^1 f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) / f_* \text{End}(E))[-1]. \end{aligned}$$

Combined with Lemma 5.3(ii) this proves the corollary. \square

Remark 5.5 One can recast the statement of Corollary 5.4 in more geometric terms. Namely, the cohomological description of Lemma 5.3(ii) leads to the following concrete description of the residual gerbe of the S -point $a_{\text{Dol}}^* \eta$ of $a_{\text{Dol}}^* T_{\pi_{\text{Dol}}}$.

Since $a_{\text{Dol}}^* \eta$ is a section of the structure morphism $a_{\text{Dol}}^* T_{\pi_{\text{Dol}}} \rightarrow S$ it follows that the coarse moduli space of the S -stack $a_{\text{Dol}}^* \eta(S)$ (taken with the reduced sub stack structure) is isomorphic to S . In particular to specify $a_{\text{Dol}}^* \eta$ one only needs to specify its residual gerbe $\mathcal{G} \rightarrow S$ [Laumon-Moret-Bailly92, Section 5]. The proof of Corollary 5.4 implies that $a_{\text{Dol}}^* T_p$ is isomorphic as a Picard stack to $h^1/h^0(f_* \text{End}(E) \rightarrow \mathbb{R}^1 f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f))$. Furthermore we have

$$\begin{aligned} \mathbb{R}^1 f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) &= \mathbb{R}^0 f_*(\sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f))[-1] \\ &= \mathbb{R}^0 f_*[\text{End}(E) \otimes \Omega_f^1 \xrightarrow{\text{ad}_{\theta_f}} \text{End}(E) \otimes \Omega_f^2 \xrightarrow{\text{ad}_{\theta_f}} \dots]. \end{aligned}$$

By examining the long exact sequence of sheaves corresponding to the distinguished triangle (5.1.9) one sees that the morphism $f_* \text{End}(E) \rightarrow \mathbb{R}^0 f_*(\sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f))[-1]$ is just the push forward of the morphism of complexes

$$\begin{array}{ccc} \text{End}(E) & \xrightarrow{\text{ad}_{\theta_f}} & \text{End}(E) \otimes \Omega_f^1 \\ \downarrow & & \downarrow \text{ad}_{\theta_f} \\ 0 & \longrightarrow & \text{End}(E) \otimes \Omega_f^2 \\ \downarrow & & \downarrow \text{ad}_{\theta_f} \\ \vdots & & \vdots \end{array}$$

Note moreover that since $\theta_f \in H^0(X, \text{End}(E) \otimes \Omega_f^1)$ is in the kernel of ad_{θ_f} , the push forward of θ_f can be viewed as a global section of the sheaf $\mathbb{R}^0 f_*(\sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f))[-1]$ on S .

We will abuse notation and write $\text{ad}_{\theta_f} : f_* \text{End}(E) \rightarrow \mathbb{R}^0 f_*(\sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f))[-1]$ and $\theta_f \in H^0(S, \mathbb{R}^0 f_*(\sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f))[-1])$.

The sections of $a_{\text{Dol}}^* T_p$ over an S -scheme $T \rightarrow S$ are pairs (A, α) where $A \rightarrow T$ is a $f_{T*} \text{End}(E_T)$ torsor and $\alpha : A \rightarrow \mathbb{R}^0 f_{T*}(\sigma_{\geq 1} \mathfrak{g}_{f_T}(E_T, \theta_{f_T}))[-1])$ is $\text{ad}_{\theta_{f_T}}$ -equivariant map. By Lemma 5.3(ii) it follows that $a_{\text{Dol}}^* \eta$ is represented by the pair $(f_* \text{End}(E), \theta_f + \text{ad}_{\theta_f}(\bullet))$. In particular the residual gerbe \mathcal{G} of $a_{\text{Dol}}^* \eta$ is just the neutral gerbe $B(\ker(\text{ad}_{\theta_f}))$ on S .

It is instructive to compare the statement of Corollary 5.4 with this geometric description. Given a point $s \in S$ put $E_s := E|_{X_s}$ and $\theta_s := \theta_f|_{X_s}$. Let as before B denote the nilpotent scheme $\text{Spec}(\mathbb{C}[t]/(t^2))$ and let $p_{X_s} : X_s \times B \rightarrow X_s$ and $p_B : X_s \times B \rightarrow B$ denote the natural projections. Identify B with the first infinitesimal neighborhood of $1 \in \mathbb{C}^\times$, i.e. put $t = \lambda - 1$ where λ is a standard coordinate on \mathbb{C}^\times , i.e. $\mathbb{C}^\times = \text{Spec}(\mathbb{C}[\lambda, \lambda^{-1}])$. The point $\epsilon_f(s) = (E_s, \theta_s) \in \mathcal{M}_{\text{Dol}}(X_s, n)$ will be fixed under the \mathbb{C}^\times -action iff there is an automorphism $\psi : p_{X_s}^* E \xrightarrow{\sim} p_{X_s}^* E$ satisfying: (a) $\psi|_{X_s \times \text{Spec}(\mathbb{C})} = \text{id}_{E_s}$; and (b) $\psi \circ p_{X_s}^* \theta_s = (\lambda p_{X_s}^* \theta_s) \circ \psi$, where by abuse of notation λ denotes the 1-jet of the function λ at $1 \in \mathbb{C}^\times$.

Now due to (a) and (b) we can write $\psi = \text{id} + t\varphi$ with $\varphi \in \text{End}(E_s)$ and $\lambda p_{X_s}^* \theta_s = \theta_s + t\theta_s$ and hence $\epsilon_f(s) \in \mathcal{M}_{\text{Dol}}(X_s, n)^{\mathbb{C}^\times}$ iff $(\text{id} + t\varphi) \circ \theta_s = (\theta_s + t\theta_s) \circ (\text{id} + t\varphi)$, i.e. if and only if $\theta_s + \text{ad}_{\theta_s}(\varphi) = 0$. This of course is exactly the statement of Corollary 5.4

We are now ready to prove Theorem A. With the hope of making the exposition more accessible we first treat the case when $f : X \rightarrow S$ is of relative dimension one and the section $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$ passes only through smooth points of moduli, i.e. when $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}^o(X/S, n) \subset \mathcal{M}_{\text{DR}}(X/S, n)$.

5.2 A warmup - the smooth case

For the duration of this section assume that S is a smooth quasi-projective variety and that $f : X \rightarrow S$ is a smooth family of integral curves over S . Let $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}^o(X/S, n)$ be an algebraic section which is horizontal with respect to the non-abelian Gauss-Manin connection. Let (F, ∇_f) be a relative connection on X representing a_{DR} . Proposition 4.6 implies that there exists a global integrable connection (F, ∇) which induces ∇_f along the fibers of f . By replacing if necessary (F, ∇) by its semi simplification (cf. [Simpson92, Lemma 3.5]) we may assume without a loss of generality that (F, ∇) is a reductive local system on X . Therefore the statement of Theorem A becomes equivalent to the following

Proposition 5.6 *Let $f : X \rightarrow S$ be a smooth family of integral curves over a projective base S . Let (F, ∇) be a reductive local system on X and assume that there exists a point $o \in S$ so that the induced local system (F_o, ∇_o) on X_o underlies a $\mathbb{C}VHS$ and such that (F_o, ∇_o) is a smooth point of $\mathcal{M}_{\text{DR}}(X_o, n)$. Then for any $s \in S$ the local system (F_s, θ_s) underlies a $\mathbb{C}VHS$ on X_s .*

Proof. Let $s \in S$ be any point. By the Lefschetz hyperplane section theorem we can always find a smooth curve $C \subset S$ cut out by hyperplanes of sufficiently high degree so that C contains the two points $o, s \in S$. Since the horizontality with respect to the non-abelian Gauss-Manin connection as well as the \mathbb{C}^\times -action on the Dolbeault moduli spaces are stable

under base change we may assume without losing generality that S is a smooth projective curve.

Let (E, θ) be the Higgs bundle on X corresponding to (F, ∇) and let (E, θ_f) be the induced relative Higgs bundle. Consider the section

$$\epsilon_f \in H^0(S, \mathbb{R}^1 f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) / f_* \text{End}(E))$$

defined after the proof of Lemma 5.3. From the long exact sequence of sheaves corresponding to the distinguished triangle (5.1.9) it follows that $\mathbb{R}^1 f_* \sigma_{\geq 1} \mathfrak{g}_f(E, \theta_f) / f_* \text{End}(E) \subset \mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f)$ and so ϵ_f can be thought of as an element in $H^0(S, \mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f))$.

Since (F, ∇) represents a horizontal section of the local system of stacks $\mathcal{M}_{\text{DR}}(X/S, n) \rightarrow S$ which passes through a smooth point of $\mathcal{M}_{\text{DR}}(X_o, n)$ it follows that (F_s, ∇_s) will be a smooth point of $\mathcal{M}_{\text{DR}}(X_s, n)$ for all s . Hence, according to Corollary 5.4 the proposition will be proven if we show that $\epsilon_f = 0$ in $H^0(S, \mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f))$. By hypothesis (E_o, θ_o) is \mathbb{C}^\times -fixed and so by Corollary 5.4 we have $\epsilon_f(o) = 0$. On the other hand the sheaf $\mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f)$ can be naturally extended to a Higgs sheaf on S . Indeed consider first direct image [Simpson93, Section 4] of the Higgs bundle $(\text{End}(E), \text{ad}_\theta)$ under the projective map $f : X \rightarrow S$. To describe this direct image note that since f is a fibration of curves over a curve we have

$$\begin{aligned} \mathfrak{g}_f(E, \theta_f) &= [\text{End}(E) \xrightarrow{\text{ad}_{\theta_f}} \text{End}(E) \otimes \Omega_f^1] \\ \mathfrak{g}_X(E, \theta) &= [\text{End}(E) \xrightarrow{\text{ad}_\theta} \text{End}(E) \otimes \Omega_X^1 \xrightarrow{\text{ad}_\theta} \text{End}(E) \otimes \Omega_X^2] \end{aligned}$$

and hence we have a short exact sequence of complexes on X

$$(5.2.10) \quad 0 \rightarrow \mathfrak{g}_f(E, \theta_f)[-1] \otimes f^* \Omega_S^1 \rightarrow \mathfrak{g}_X(E, \theta) \rightarrow \mathfrak{g}_f(E, \theta_f) \rightarrow 0$$

As explained in [Simpson93, Section 4] the first direct image of $(\text{End}(E), \text{ad}_\theta)$ under f is just the sheaf $\mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f)$ together with the first edge homomorphism $\delta : \mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f) \rightarrow \mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f) \otimes \Omega_S^1$ for the long exact sequence of hyper-derived images corresponding to (5.2.10).

Recall next that ϵ_f was defined as the image of $R^1 f_* \theta_f$ in $\mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f)$. On the other hand we have a natural element $\epsilon \in H^0(S, \mathbb{R}^1 f_* \mathfrak{g}_X(E, \theta))$ defined analogously as the image of $R^1 f_* \theta$. Since θ restricts to the relative Higgs field θ_f it follows that ϵ maps to ϵ_f under the natural map $\mathbb{R}^1 f_* \mathfrak{g}_X(E, \theta) \rightarrow \mathbb{R}^1 f_* \mathfrak{g}_f(E, \theta_f)$. Therefore from the long exact sequence of hyper-derived images corresponding to (5.2.10) we conclude that $\delta(\epsilon_f) = 0$, i.e. ϵ_f is in fact an element in the 0-th cohomology group $H^0(S, (\mathbb{R}^1 f_* \mathfrak{g}_X(E, \theta), \delta))$ of the Higgs sheaf $(\mathbb{R}^1 f_* \mathfrak{g}_X(E, \theta), \delta)$.

To finish the proof of the proposition it remains only to note that by [Simpson93, Corollary 5.2] the Higgs bundle $(\mathbb{R}^1 f_* \mathfrak{g}_X(E, \theta), \delta)$ on S corresponds to the local system $(\mathbb{R}^1 f_* \mathcal{E}_f(F, \nabla_f), D)$ with $D : \mathbb{R}^1 f_* \mathcal{E}_f(F, \nabla_f) \rightarrow \mathbb{R}^1 f_* \mathcal{E}_f(F, \nabla_f) \otimes \Omega_S^1$ being just the first edge homomorphism in the long exact sequence of hyper-derived images corresponding to

$$(5.2.11) \quad 0 \rightarrow \mathcal{E}_f(F, \nabla_f)[-1] \otimes \Omega_S^1 \rightarrow \mathcal{E}_X(F, \nabla) \rightarrow \mathcal{E}_f(F, \nabla_f) \rightarrow 0.$$

Combined with Simpson formality result [Simpson92, Lemma 1.2] this gives

$$\epsilon \in H^0(S, (\mathbb{R}^1 f_* \mathfrak{g}_X(E, \theta), \delta)) = H^0(S, (\mathbb{R}^1 f_* \mathcal{E}_f(F, \nabla_f), D)).$$

In particular ϵ_f can be interpreted as an algebraic section of the sheaf $\mathbb{R}^1 f_* \mathcal{E}_f(F, \nabla_f)$ which is horizontal with respect to D . Hence if ϵ vanishes at a point it vanishes everywhere which proves the proposition. \square

Remark 5.7 (i) The proof of the proposition works in essentially the same way without reducing to the case when S is a curve. Indeed to make the above argument work in general one only needs to replace (5.2.10) (respectively (5.2.11)) by the short exact sequence of complexes

$$0 \rightarrow \mathfrak{g}_f(E, \theta_f)[-1] \otimes f^* \Omega_S^1 \rightarrow \mathfrak{g}_X(E, \theta)/I^2 \mathfrak{g}_X(E, \theta) \rightarrow \mathfrak{g}_f(E, \theta_f) \rightarrow 0,$$

where as usual

$$\begin{aligned} I^1 \mathfrak{g}_X(E, \theta) &:= \text{im}[\mathfrak{g}_X(E, \theta) \otimes f^* \Omega_S \rightarrow \mathfrak{g}_X(E, \theta)] \\ I^2 \mathfrak{g}_X(E, \theta) &:= \text{im}[I^1 \mathfrak{g}_X(E, \theta) \otimes f^* \Omega_S \rightarrow \mathfrak{g}_X(E, \theta)] \end{aligned}$$

(see [Simpson93, Section 4]).

(ii) The hypothesis that f is of relative dimension one is superfluous in the statement of Proposition 5.6 and was only used to simplify the exposition. The proof for the case of general fiber dimension works in exactly the same way. The reason we chose to state Proposition 5.6 for fiber dimension one only will become clear in the next section where we show that the smoothness of the morphisms of π_{DR} and π_{Dol} is easier to control when one is dealing with curves.

The proof of Proposition 5.6 has the following immediate corollary

Corollary 5.8 *Let $f : X \rightarrow S$ be a smooth projective morphism with connected fibers to a smooth quasi-projective S . Assume that f extends to a projective morphism $\bar{f} : \bar{X} \rightarrow \bar{S}$ where \bar{X} and \bar{S} are smooth and projective and $\bar{S} \setminus S$ and $\bar{X} \setminus X$ are divisors with strict normal crossings. Let (F, ∇) be a reductive local system on \bar{X} and assume that there exists a point $o \in S$ so that the induced local system (F_o, ∇_o) on X_o underlies a $\mathbb{C}VHS$. Then for any $s \in S$ the local system (F_s, θ_s) underlies a $\mathbb{C}VHS$ on X_s .*

Proof. The proof is essentially the same but at the last stage we need to invoke a much stronger theorem of Simpson [Simpson93, Corollary 5.12] which asserts that the filtered regular Higgs bundle (= parabolic Higgs bundle) $(\mathbb{R}^1 f_* \mathfrak{g}_X(E, \theta), \delta)$ on S corresponds to the local system $(\mathbb{R}^1 f_* \mathcal{E}_f(F, \nabla_f), D)$. \square

Remark 5.9 (i) Note that the reduction to the situation where S is a curve is really essential here since the theory of regular filtered Higgs bundles [Simpson90] is developed only in the curve case.

(ii) It is tempting to try and extend the previous corollary to a situation where (F, ∇) is a reductive local system on \overline{X} with logarithmic poles along $\overline{X} \setminus X$.

There are several obstructions to carrying out the arguments in this case. First of all it is unclear whether such a local system will have a harmonic metric. There are however some existence results under mild additional assumptions. For example [Biquard97, Theorem 11.4] guarantees the existence of a harmonic metric for (F, ∇) over X as long as (F, ∇) is stable as a parabolic local system and the residual primitive homogeneous local systems on the total space of the normal bundle of $\overline{X} \setminus X$ are semi simple. Similarly in [Corlette92] the existence of a harmonic metric on (F, ∇) is proven under the assumptions that (F, ∇) is reductive and has a quasi-unipotent monodromy at infinity. Ideally one would like to apply these results to a logarithmic (F, ∇) representing a section a_{DR} which is horizontal for the non-abelian Gauss-Manin connection. For this one will have to analyze the behavior of a_{DR} at infinity which is an interesting question in its own right.

The second problem is in the lack of a push forward result similar to [Simpson93, Corollary 5.12] which works for parabolic local systems with poles along $\overline{X} \setminus X$. Specifically one needs to find the right growth conditions for sections in a harmonic bundle near a point in a normal crossings divisor.

5.3 A reduction - the general case

In this section we show how to reduce the statement Theorem A to the situation in Proposition 5.6. For this we have to explain how to: (a) deal with morphisms $f : X \rightarrow S$ of fiber dimension bigger than one; and (b) how to deal with sections $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$ that do not necessarily land in $\mathcal{M}_{\text{DR}}^o(X/S, n)$.

The problem (a) is quite mild. In fact Remark 5.7(ii) shows how (a) can be tackled directly within the method of proof of Proposition 5.6. Alternatively we may use the Lefschetz hyperplane section theorem and its non-abelian version - the Mehta-Ramanathan type restriction result [Simpson92, Proposition 3.6] (see also [Huybrechts-Lehn97, Theorem 7.2.1]). This second route is preferable since it will put us into a favorable setup for dealing with (b).

Concretely let $f : X \rightarrow S$ be a smooth projective morphism with connected fibers. Assume that S is quasi-projective and let $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$ be an algebraic section which is horizontal with respect to the non-abelian Gauss-Manin connection. Since the statement of Theorem A is stable under base changes $T \rightarrow S$ with T smooth and projective we may assume without a loss of generality that $f : X \rightarrow S$ has a section $\xi : S \rightarrow X$.

Let $i : C \hookrightarrow X$ be a general enough intersection of relative hyperplanes so that $f \circ i : C \rightarrow S$ is smooth with connected fibers and $\dim_{\mathbb{C}}(C/S) = 1$. By the Lefschetz hyperplane section theorem we have a surjection $\pi_1(C_s) \rightarrow \pi_1(X_s)$ for every $s \in S$ and therefore i induces a closed immersion of algebraic stacks $\mathcal{M}_{\text{B}}(X/S, n) \subset \mathcal{M}_{\text{B}}(C/S, n)$. The strong form of the

Riemann-Hilbert correspondence [Simpson95, Proposition 7.8] (we need the existence of ξ for that!) yields then a closed immersion of analytic stacks $\mathcal{M}_{\text{DR}}(X/S, n)^{\text{an}} \subset \mathcal{M}_{\text{DR}}(C/S, n)^{\text{an}}$ which in turn implies that the natural morphism $i_{\text{DR}}^* \mathcal{M}_{\text{DR}}(X/S, n) \rightarrow \mathcal{M}_{\text{DR}}(C/S, n)$ of Artin algebraic stacks is also a closed immersion. Furthermore due to the functoriality of the algebraic construction of the non-abelian Gauss-Manin connection the morphism i_{DR}^* must be horizontal and so $i_{\text{DR}}^* \circ a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}(C/S, n)$ is an algebraic section which is horizontal with respect to the non-abelian Gauss-Manin connection as well. Next observe that the surjectivity of $\pi_1(C_s) \rightarrow \pi_1(X_s)$ puts us in a position to apply [Simpson92, Corollary 4.3] and conclude that $a_{\text{DR}}(s)$ underlies a $\mathbb{C}\text{VHS}$ if and only if $i_{\text{DR}}^* \circ a_{\text{DR}}(s)$ underlies a $\mathbb{C}\text{VHS}$.

The next fact we need is that due to [Simpson92, Proposition 3.6] the semistability of Higgs bundles is preserved by restrictions to hyperplane sections. Hence in the same way as in the proof of Lemma 5.2 we can use [Simpson92, Corollary 3.10] to reduce Theorem A to the following

Lemma 5.10 *Let $f : X \rightarrow S$ be a smooth morphism with connected fibers of relative dimension one. Assume that S is quasi-projective and that f admits a section ξ . Let $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$ be a section represented by a relative connection (F, ∇_f) . Then*

- (a) *a_{DR} is horizontal iff there exists a global connection $\nabla : F \rightarrow F \otimes \Omega_X^1$ which induces ∇_f .*
- (b) *if S is projective, then (F_s, ∇_s) underlies a $\mathbb{C}\text{VHS}$ iff $\epsilon_f(s) = 0$*

Proof. If a_{DR} happens to map S into $\mathcal{M}_{\text{DR}}^o(X/S, n)$, then this is the content of Proposition 4.6. For a general a_{DR} we need to analyze the singularities of the morphism π_{DR} .

Let $f : X \rightarrow S$ be a smooth fibration of curves of genus $g > 1$. Let G be a complex reductive group. It is well known (see e.g. [Behrend91, Corollaries 4.5.2 and 8.1.9]) that the stack $\mathcal{Bun}(X/S, G)$ is a smooth stack over S of relative dimension $(g - 1) \dim G$. In particular $\mathcal{Bun}^o(X/S, n) = \mathcal{Bun}(X/S, n)$. Moreover from the definition of $\mathcal{M}_{\text{Dol}}(X/S, n)$ and the fact that $\dim(X/S) = 1$ it is clear that $\mathcal{M}_{\text{Dol}}(X/S, n)$ can be identified with the relative cotangent stack $T_\pi^\vee \rightarrow \mathcal{Bun}(X/S, n)$ of $\pi : \mathcal{Bun}(X/S, n) \rightarrow S$. Here by T_π^\vee one means the vector bundle stack $\text{Spec}(S^\bullet R^1 \text{pr}_* \text{End}(E_{\text{un}}))$ where as usual $E_{\text{un}} \rightarrow \mathcal{Bun}(X/S, n) \times_S X$ is the universal bundle and $\text{pr} : \mathcal{Bun}(X/S, n) \times_S X \rightarrow \mathcal{Bun}(X/S, n)$ is the natural projection.

This indicates that it is not unreasonable to expect that $\mathcal{M}_{\text{Dol}}(X/S, n)$ (and hence $\mathcal{M}_{\text{DR}}(X/S, n)$) will be close enough to being smooth. In fact it is easy to see that the stack $\mathcal{M}_{\text{DR}}(X/S, n)$ (respectively $\mathcal{M}_{\text{Dol}}(X/S, n)$) embeds in a stack which is smooth over S . To construct such an embedding one uses the following well known (see for example [Beilinson-Drinfeld99, Section 2.11]) rigidification trick.

Let $\mathcal{M}_{\text{DR}}(X/S(\log \xi), n)$ be the stack parameterizing relative local systems on $f : X \rightarrow S$ with logarithmic poles along $\xi(S)$. Let $(F_{\text{un}}, \nabla_{\text{un}}) \rightarrow \mathcal{M}_{\text{DR}}(X/S(\log \xi), n) \times_S X$ be the universal relative local system and let $\mathcal{F} := (\text{id} \times \xi)^* F_{\text{un}}$. The residue of ∇_{un} along $\xi(S)$ is a section $\text{Res}(\nabla_{\text{un}}) \in H^0(\mathcal{M}_{\text{DR}}(X/S(\log \xi), n), \text{End}(\mathcal{F}))$ and $\mathcal{M}_{\text{DR}}(X/S, n)$ is just the closed substack of $\mathcal{M}_{\text{DR}}(X/S(\log \xi), n)$ cut out by the equation $\text{Res}(\nabla_{\text{un}}) = 0$. Furthermore the maximal substack $\mathcal{M}_{\text{DR}}^o(X/S(\log \xi), n) \subset \mathcal{M}_{\text{DR}}(X/S(\log \xi), n)$ which is smooth over S can

be described explicitly in this case. Indeed, note first that the same argument as in the proof of Lemma 4.2 shows that the deformation-obstruction complex for $\mathcal{M}_{\text{DR}}(X/S(\log \xi), n)$ is just the complex

$$(5.3.12) \quad \text{End}(F_{\text{un}}) \xrightarrow{\text{ad}_{\nabla_{\text{un}}}} \text{End}(F_{\text{un}}) \otimes \text{pr}_X^* \Omega_f^1(\log \xi(S)).$$

Consequently if we put $f_{\text{DR}} : \mathcal{M}_{\text{DR}}(X/S(\log \xi), n) \times_S X \rightarrow \mathcal{M}_{\text{DR}}(X/S(\log \xi), n)$ for the natural projection we can characterize $\mathcal{M}_{\text{DR}}^o(X/S(\log \xi), n)$ as the open sub stack over which the morphism of coherent sheaves

$$R^1 f_{\text{DR}*} \text{End}(F_{\text{un}}) \xrightarrow{R^1 f_{\text{DR}*} \text{ad}_{\nabla_{\text{un}}}} R^1 f_{\text{DR}*} (\text{End}(F_{\text{un}}) \otimes \text{pr}_X^* \Omega_f^1(\log \xi(S)))$$

is surjective. In particular $\mathcal{M}_{\text{DR}}^o(X/S(\log \xi), n)$ is smooth of dimension $(2g-1)n^2$ over S . Moreover if $T \rightarrow S$ is an S -scheme and if (F, ∇_{f_T}) is in $\mathcal{M}_{\text{DR}}(X/S(\log \xi), n)(T)$ observe that by relative duality (F, ∇_{f_T}) will belong to $\mathcal{M}_{\text{DR}}^o(X/S(\log \xi), n)(T)$ iff the morphism

$$\text{ad}_{\nabla_{f_T}} : f_{T*}(\text{End}(F)(-\xi_T(T))) \rightarrow f_{T*}(\text{End}(F) \otimes \Omega_{f_T}^1)$$

is injective.

On the other hand if (F, ∇_{f_T}) is in $\mathcal{M}_{\text{DR}}(X/S, n)(T)$ to begin with, then a section $s \in \Gamma(U, f_{T*}(\text{End}(F)(-\xi_T(T))))$ for some open $U \subset S$ will be in the kernel of $\text{ad}_{\nabla_{f_T}}$ if and only if s is a ∇_T -horizontal section of $\text{End}(F)$ which vanishes along $\xi_T(T)$, i.e. if and only if $s = 0$. This shows that $\mathcal{M}_{\text{DR}}(X/S, n)$ is in fact a sub stack of $\mathcal{M}_{\text{DR}}^o(X/S(\log \xi), n)$.

Observe next that the family $\mathcal{M}_{\text{DR}}(X/S(\log \xi), n) \rightarrow S$ also has an algebraic integrable connection which can be defined in the same way as the non-abelian Gauss-Manin connection in terms of formal groupoids [Simpson97a, Section 8]. Also the fact that we have an inclusion of formal groupoids $X_{\text{DR}} \subset X_{\text{DR}}(\log \xi(S))$ (see e.g. Remark 2.12 for notation) implies that $\mathcal{M}_{\text{DR}}(X/S, n) \subset \mathcal{M}_{\text{DR}}^o(X/S(\log \xi), n)$ is an inclusion of crystals of stacks. Thus $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}^o(X/S(\log \xi), n)$ is a horizontal section.

Finally the fact that (5.3.12) is the deformation obstruction complex for the stack $\mathcal{M}_{\text{DR}}(X/S(\log \xi), n)$ combined with the smoothness of $\mathcal{M}_{\text{DR}}^o(X/S(\log \xi), n)$ puts us in a situation where the arguments we used to prove Proposition 4.6 work verbatim. This shows that ∇_f comes from a global logarithmic connection $\nabla : F \rightarrow F \otimes \Omega_X^1(\log \xi(S))$. But $\xi(S)$ is transversal to the fibers of f and so $\text{Res}(\nabla) = \text{Res}(\nabla_f) = 0$. This proves part (a) of the lemma.

Similarly to prove part (b) we have to view the global Higgs field (E, θ) corresponding to (F, ∇) as a section in $\mathcal{M}_{\text{Dol}}^o(X/S(\log \xi), n)$. Again we can identify the deformation obstruction complex of $\mathcal{M}_{\text{Dol}}^o(X/S(\log \xi), n)$ as the complex

$$\text{End}(E_{\text{un}}) \xrightarrow{\text{ad}_{\theta_{\text{un}}}} \text{End}(E_{\text{un}}) \otimes \text{pr}_X^* \Omega_f^1(\log \xi(S)),$$

and the same reasoning as in the proof of Lemma 5.3(ii) and Corollary 5.4 shows that (E_s, θ_s) will be \mathbb{C}^\times -fixed only when the section

$$\epsilon_f \in H^0(S, f_*(\text{End}(E) \otimes \Omega_f^1)/f_* \text{End}(E)) \subset H^0(S, f_*(\text{End}(E) \otimes \Omega_f^1(\log \xi(S)))/f_* \text{End}(E))$$

vanishes at s . The lemma is proven. □

Remark 5.11 (i) It is shown in [Beilinson-Drinfeld99, Proposition 2.11.2] that in the assumptions of the lemma the morphism $\mathcal{M}_{\text{DR}}(X/S, n) \rightarrow S$ is a l.c.i. morphism of dimension $(2g - 2)n^2 + n$.

(ii) The rigidification trick used in the proof of the previous lemma is not really necessary and in fact by using crystals of 2-stacks one should be able to prove Proposition 4.6(1) for arbitrary sections $a_{\text{DR}} : S \rightarrow \mathcal{M}_{\text{DR}}(X/S, n)$.

Appendix A Tangent stacks

The language of algebraic stacks is the natural framework for describing moduli problems in algebraic geometry. It grew out of M. Artin's approach to moduli [Artin69, Artin70] and is by now a standard tool in deformation theory. Since the only comprehensive treatment of the theory of Artin algebraic stacks is the Orsay preprint [Laumon-Moret-Bailly92] we will review briefly the definition and the basic properties of the tangent stack of an algebraic stack. Our main references are [Artin74], [Laumon-Moret-Bailly92] and [Vistoli89, Appendix].

A.1 Algebraic stacks

The main problem one encounters in constructing a moduli space parameterizing a given family of geometric objects is the problem of representability. Very often the set of equivalence classes of our objects is too wild and does not carry any natural geometric structure. Typically the main obstacle for finding such a structure is the different size of the equivalence classes. To remedy that one tries to retain somehow the information about the many representatives of a given equivalence class. Thus one is naturally lead to replace the set of equivalence classes by the category of all of their representatives. This category has a rather special nature since the morphisms between any two objects come from an equivalence relation and are therefore all invertible. Categories for which all morphisms are invertible are called *groupoids*. In Artin's approach to moduli the first step is to try and represent a given moduli problem not by a scheme but by a category comprised of groupoids endowed with extra geometric structure. When the formal properties of such a geometric structure are written down one gets the notion of an algebraic (or Artin) stack. This is very similar to the process of putting a scheme structure on set with the key difference that the points in the stack are objects in a category (and hence have intrinsically defined automorphisms) rather than elements in a set. The properties one needs in order to do geometry on a groupoid had crystallized in the seminal works of Mumford [Mumford65], Deligne and Mumford [Deligne-Mumford69] and Artin [Artin74]. Even though the actual definition given

below is rather formal, it is concrete enough to allow us to operate with an algebraic stack in the same way as with any other object in algebraic geometry.

We will need the notion of a stack over a base scheme S . Denote by (Sch/S) the category of schemes over S . As explained above intuitively one should think of a stack as a collection of groupoids endowed with geometric structure. All groupoids form a 2-category (Grp) . The objects of (Grp) are the groupoids, the 1-morphisms are the functors between groupoids and the 2-morphisms are the isomorphisms of functors between groupoids.

Definition A.1 *A groupoid over S (or a pre-sheaf of groupoids over S) is a lax functor*

$$\mathcal{X} : (\text{Sch}/S)^{\text{op}} \rightarrow (\text{Grp}).$$

In other words \mathcal{X} assigns a groupoid \mathcal{X}_U to any S -scheme U (the groupoid of sections of \mathcal{X} over U), a change of base functor $\varphi^ : \mathcal{X}_V \rightarrow \mathcal{X}_U$ to any morphism $\varphi : U \rightarrow V$ of S -schemes and a canonical isomorphism of functors*

$$\psi^* \circ \varphi^* \cong (\varphi \circ \psi)^*$$

for any $W \xrightarrow{\psi} V$ and $V \xrightarrow{\varphi} U$ - composable arrows in (Sch/S) . Furthermore these canonical isomorphisms have to satisfy the standard cocycle condition.

Some good examples to keep in mind are:

Example A.2 (i) Every algebraic space (cf. [Knutson71]) X over S has an associated S -groupoid which we will again denote by X . For any $U \in \text{Ob}(\text{Sch}/S)$ the groupoid X_U is just the discrete category corresponding to the set of U points $\text{Hom}(U, X)$ of X . For any morphism $V \xrightarrow{\varphi} U$ the change of base morphism φ^* is just the restriction $\text{Hom}(U, X) \rightarrow \text{Hom}(V, X)$.

(ii) Let G be an affine group scheme over S . Let X be an algebraic space over S equipped with a G -action. We have the quotient groupoid $[X/G] : (\text{Sch}/S)^{\text{op}} \rightarrow (\text{Grp})$ for which $[X/G]_U$ is the groupoid of all pairs (P, a) where P is a G -torsor on U and $a : P \rightarrow X \times_S U$ is a G -equivariant morphism.

As special case is to take $X = S$ equipped with the trivial G action. The quotient $[S/G]$ is just the groupoid of all G -torsors over S . It is called the *classifying groupoid* of G and is usually denoted by BG .

(iii) Let X be an algebraic space over S . The S -groupoid $\text{Qcoh}_{X/S}$ of quasi-coherent \mathcal{O}_X -modules is defined as follows. For any $U \rightarrow S$ the groupoid of sections of $\text{Qcoh}_{X/S}$ over U is the category whose objects are all quasi-coherent sheaves on $X \times_S U$ which are flat over U and whose morphisms are the isomorphisms of quasi-coherent sheaves. For any morphism $V \xrightarrow{\varphi} U$ in (Sch/S) the change of base functor φ^* is just the pull-back via $\text{id}_X \times_S \varphi$.

(iv) Fix a reductive group G . Let $f : X \rightarrow S$ be smooth projective and let $x : S \rightarrow X$ be a section. Denote by $R_B(X/S, x, G)$ the family of representation spaces of the fundamental groups of the fibers of f . In other words $R_B(X/S, x, G)_s := \text{Hom}(\pi_1(X_s, x(s)), G)$.

Let $R_{\text{DR}}(X/S, x, G)$ denote the fine moduli scheme of principal G -bundles on X with a relative integrable connection and a frame over x constructed in [Simpson94]. Finally put $R_{\text{Dol}}(X/S, x, G)$ for the fine moduli scheme of relative semistable principal Higgs bundles with vanishing rational Chern classes and a frame over x constructed in [Simpson94]. The group G acts on all of these spaces and by taking quotients we get pre-sheaves of groupoids $\mathcal{M}_{\text{B}}(X/S, G) = [R_{\text{B}}(X/S, x, G)/G]$, $\mathcal{M}_{\text{DR}}(X/S, G) = [R_{\text{DR}}(X/S, x, G)/G]$ and $\mathcal{M}_{\text{Dol}}(X/S, G) = [R_{\text{Dol}}(X/S, x, G)/G]$ corresponding to the moduli problems for representations, local systems and Higgs bundles respectively.

In general, when dealing with moduli, one starts with some class \mathfrak{X} of geometric objects over S (e.g. schemes, sheaves, maps, etc.) and an equivalence relation “ \sim ” on \mathfrak{X} . Next one tries to represent the functor to sets

$$\begin{aligned} (\text{Sch}/S)^{\text{op}} &\xrightarrow{X^\natural} (\text{Set}) \\ (U \rightarrow S) &\longrightarrow \left\{ \begin{array}{l} \text{the set of equivalence classes of fam-} \\ \text{ilies in } \mathfrak{X} \text{ parameterized by } U \end{array} \right\} \end{aligned}$$

by a S -scheme. This usually fails since X^\natural is rarely a sheaf in any reasonable topology and so cannot be representable. On the other hand, very often the pre-sheaf of groupoids

$$\begin{aligned} (\text{Sch}/S)^{\text{op}} &\xrightarrow{\mathcal{X}} (\text{Grp}) \\ (U \rightarrow S) &\longrightarrow \left\{ \begin{array}{l} \text{the groupoid with objects - the fam-} \\ \text{ilies in } \mathfrak{X} \text{ parameterized by } U \text{ and} \\ \text{with morphisms - the equivalences} \\ \text{of families} \end{array} \right\} \end{aligned}$$

is a sheaf. Moreover X^\natural is easily recovered from \mathcal{X} since for every $U \in \text{Ob}(\text{Sch}/S)$ the set $X^\natural(U)$ is just the set of connected components of the groupoid $\mathcal{X}(U)$. For example the moduli functors $M_{\text{B}}^\natural(X/S, n)$, $M_{\text{DR}}^\natural(X/S, n)$, $M_{\text{Dol}}^\natural(X/S, n)$ from [Simpson94] are obtained in this way from the pre-sheaves of groupoids $\mathcal{M}_{\text{B}}(X/S, G)$, $\mathcal{M}_{\text{DR}}(X/S, G)$ and $\mathcal{M}_{\text{Dol}}(X/S, G)$ respectively. The general principle is that instead of trying to represent X^\natural by a scheme we may try to put enough geometric structure on \mathcal{X} so that it can be treated as a scheme.

Roughly speaking the stacks are pre-sheaves of groupoids which become sheaves when considered in an appropriate topology.

Definition A.3 *Let \mathcal{X} be a groupoid over S . The pre-sheaf \mathcal{X} is called a stack in the fppf/smooth/étale topology if*

- (i) *For any U in (Sch/S) and any two objects x, y in $\mathcal{X}(U)$ the pre-sheaf*

$$\begin{aligned} (\text{Sch}/U)^{\text{op}} &\xrightarrow{\text{Isom}(x, y)} (\text{Set}) \\ (V \rightarrow U) &\longrightarrow \text{Hom}_{\mathcal{X}_V}(x_V, y_V) \end{aligned}$$

is a sheaf in the fppf/smooth/étale topology.

- (ii) If $\{V_i \xrightarrow{\varphi_i} U\}$ is a covering of $U \in \text{Ob}(\text{Sch}/S)$ in the fppf/smooth/étale topology and if (x_i, f_{ij}) is a descend datum (that is - $x_i \in \text{Ob } \mathcal{X}_{V_i}$ and $f_{ji} : x_i|_{V_{ji}} \xrightarrow{\cong} x_j|_{V_{ji}}$ are morphisms in $\mathcal{X}_{V_{ji}}$ satisfying the cocycle condition) relative to $\{V_i \xrightarrow{\varphi_i} U\}$, then (x_i, f_{ij}) is effective, i.e. - there exists an object x in \mathcal{X}_U and isomorphisms $f_i : x|_{V_i} \xrightarrow{\cong} x_i$ in \mathcal{X}_{V_i} so that for every i, j one has $f_j|_{V_{ji}} = f_{ji} \circ (f_i|_{V_{ji}})$.

It is not hard to verify that all of the S -groupoids in example A.2 are actually stacks. Checking that the pre-sheaves from Example A.2 (i) (ii) and (iv) are stacks is straightforward. The proof that Example A.2 (iii) is a stack can be found in [SGA1, Sections 1.1 and 1.2 of Exposé VIII].

Definition A.4 A morphism between two stacks over S is just a 1-morphisms of pre-sheaves of groupoids. A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is injective if for every $U \in (\text{Sch}/S)$ the functor $f_U : \mathcal{X}_U \rightarrow \mathcal{Y}_U$ is faithful. A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is surjective if for every $U \in (\text{Sch}/S)$ and every $y \in \text{Ob } \mathcal{Y}_U$ there exists a covering family $\{V \rightarrow U\}$ in $(\text{Sch}/S)_{\text{fppf/smooth/étale}}$ and a $x \in \mathcal{X}_V$ for which $f_V(x)$ is isomorphic to y in \mathcal{Y}_V .

In order to do geometry on a stack \mathcal{X} it is essential to be able to patch geometric data that is defined “locally” on \mathcal{X} . For this one needs a notion of a fiber product of stacks. Let $f : \mathcal{X} \rightarrow \mathcal{S}$ and $g : \mathcal{Y} \rightarrow \mathcal{S}$ be two morphisms of stacks. The fiber product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is the stack defined as follows. For any $U \rightarrow S$ in (Sch/S) the objects of $(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})(U)$ are triples (x, y, α) where $x \in \text{Ob } \mathcal{X}(U)$, $y \in \text{Ob } \mathcal{Y}(U)$ and $\alpha : f_U(x) \rightarrow g_U(y)$ is a morphism in $\mathcal{S}(U)$. A morphism between two (x', y', α') and (x'', y'', α'') in $(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})(U)$ is a pair $(a, b) \in \text{Hom}_{\mathcal{X}(U)}(x', x'') \times \text{Hom}_{\mathcal{Y}(U)}(y', y'')$ for which $\alpha'' \circ f(a) = f(b) \circ \alpha'$. Finally for every $\varphi : V \rightarrow U$ in (Sch/S) the pull-back functor φ^* is defined component wise on every (x, y, α) and every (a, b) .

Once we have the notion of a fiber product in the 2-category of stacks we can study the local behavior of a morphism. Especially useful are morphisms between stacks which on affine scheme patches behave as morphisms of schemes.

Definition A.5 A morphism of stacks $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called representable if for any scheme U in (Sch/S) and any morphism $U \rightarrow \mathcal{Y}$ over S the fiber product $\mathcal{X} \times_{\mathcal{Y}} U$ is equivalent to an algebraic space.

Remark A.6 It is not hard to characterize the representable morphisms of stacks in purely categorical terms. Since the S -groupoids corresponding to algebraic spaces are exactly the locally discrete one it is clear that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is representable if and only if the functor $f_U : \mathcal{X}_U \rightarrow \mathcal{Y}_U$ is faithful for all U in (Sch/S) . Informally $\mathcal{X} \rightarrow \mathcal{Y}$ is representable if \mathcal{X} is less “stacky” than \mathcal{Y} .

Due to this definition any fppf local property of morphisms of schemes which is stable under base change makes sense for representable morphisms of stacks as well. More precisely

if P is such a property we will say that a representable $f : \mathcal{X} \rightarrow \mathcal{Y}$ has the property P if for every S -scheme U and every morphism $U \rightarrow \mathcal{Y}$ the morphism of algebraic spaces $\mathcal{X} \times_{\mathcal{Y}} U \rightarrow U$ has the property P . In particular we can speak of f being surjective, universally bijective, universally open or closed, separated, quasi-compact, of finite type, flat, smooth, étale, etc.

Example A.7 From the above remark it is clear that if $H \rightarrow G$ is a homomorphism of algebraic groups over a field \mathfrak{k} , then the induced morphism of stacks $BH \rightarrow BG$ is representable iff $H \rightarrow G$ is a monomorphism. In particular for $* := \mathrm{Spec}(\mathfrak{k})$ and $BG = [* / G]$ we have that $* \rightarrow BG$ is representable and that $BG \rightarrow *$ is not representable.

Now we are ready to introduce the Artin algebraic stacks. Heuristically these are stacks that look like schemes if one looks at them from the view point of the category of schemes.

Lemma-Definition A.8 *An algebraic (geometric) stack is a S -groupoid \mathcal{Z} such that*

- (1) \mathcal{Z} is a stack in the fppf/smooth/étale topology.
- (2) *One of the following equivalent conditions holds*
 - a. *The diagonal $\Delta_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathcal{Z} \times \mathcal{Z}$ is representable, separated and quasi-compact.*
 - b. *For all S -algebraic spaces X, Y and all morphisms $X \rightarrow \mathcal{Z}$ and $Y \rightarrow \mathcal{Z}$ the fiber product $X \times_{\mathcal{Z}} Y$ is equivalent to an algebraic space over S .*
 - c. *For all S -affine schemes X, Y and all morphisms $X \rightarrow \mathcal{Z}$ and $Y \rightarrow \mathcal{Z}$ the fiber product $X \times_{\mathcal{Z}} Y$ is equivalent to an algebraic space over S .*
- (3) *There exists a S -algebraic space Z and a smooth surjective morphism $p : Z \rightarrow \mathcal{Z}$. The pair (Z, p) is called an atlas of \mathcal{Z} .*

Proof. [Laumon-Moret-Bailly92, Corollary 2.12] □

Remark A.9 (i) An algebraic stack \mathcal{Z} is called a Deligne-Mumford stack if it has an atlas $p : Z \rightarrow \mathcal{Z}$ with p - étale and surjective.

(ii) An important theorem of M. Artin weakens considerably part (3) of the definition of an algebraic stack. Artin's criterion [Artin74, Theorem 6.1] asserts that a pre-sheaf of groupoids \mathcal{Z} is an algebraic stack if and only if \mathcal{Z} satisfies (1) and (2) and if there exists an algebraic space Z and a surjective fppf morphism $p : Z \rightarrow \mathcal{Z}$.

(iii) Another remarkable feature of Artin algebraic stacks is that they admit a rather concrete geometric description as a quotient of an algebraic space by a smooth equivalence relation over S . Recall [Knutson71, II, 1.1] that an equivalence relation on X is given by an algebraic S -space R together with a monomorphism $\delta : R \rightarrow X \times_S X$ so that for every S scheme U the subset $R(U) \subset X(U) \times X(U)$ is the graph of an equivalence relation of sets. The quotient of

X by the equivalence relation R is by definition the quotient sheaf (in the fppf/smooth/étale topology) of sets on (Sch/S) for the diagram

$$R \begin{array}{c} \xrightarrow{\text{pr}_1 \circ \delta} \\ \xrightarrow{\text{pr}_2 \circ \delta} \end{array} X.$$

We say that $R \rightrightarrows X$ is a *smooth* equivalence relation if the structure morphisms $\text{pr}_i \circ \delta$ are smooth and of finite type.

Given a smooth equivalence relation $R \rightrightarrows X$ we can construct not only the quotient sheaf of sets on (Sch/S) but a quotient algebraic stack $[X/R]$ as well. For any U in (Sch/S) consider the category $[X/R]'(U)$ whose objects are pairs $(V \rightarrow U, \alpha)$ where $V \rightarrow U$ is a smooth covering in (Sch/S) and $\alpha : (V \times_U V \rightrightarrows V) \rightarrow (R \rightrightarrows X)$ is a morphism of equivalence relations. For any two pairs $(V' \rightarrow U, \alpha')$ and $(V'' \rightarrow U, \alpha'')$ define

$$\text{Hom}_{[X/R]'(U)}((V' \rightarrow U, \alpha'), (V'' \rightarrow U, \alpha'')) = \left\{ \begin{array}{l} \text{the set of all isomorphisms of } f : V' \xrightarrow{\cong} V'' \\ \text{over } U \text{ for which } \alpha'' = (f \times_U f, f) \circ \alpha'. \end{array} \right\}$$

Finally for any $\varphi : U \rightarrow V$ denote by $\varphi^* : [X/R]'(V) \rightarrow [X/R]'(U)$ the natural restriction functor. Clearly $[X/R]'$ is a presheaf of groupoids on (Sch/S) . In general $[X/R]'$ is not a stack but only a pre-stack (i.e. satisfies only condition (i) in Definition A.3 but not condition (ii)). A straightforward analogue [Laumon-Moret-Bailly92, Lemma 2.2] of the usual plus construction which associates a canonical sheaf to any pre-sheaf allows us to stackify the pre-stack $[X/R]'$. Denote the resulting stack by $[X/R]$. It has an obvious smooth atlas $X \rightarrow [X/R]$ and by construction the two equivalence relations $X \times_{[X/R]} X \rightrightarrows X$ and $R \rightrightarrows X$ are canonically isomorphic. In particular $[X/R]$ is an Artin stack.

(iv) The language of smooth equivalence relations is very convenient for expressing how far a given stack is from being a scheme. For example if $R \rightrightarrows X$ is a smooth equivalence relation and if $R \rightarrow X \times_{[X/R]} X$ is unramified, then $[X/R]$ is a Deligne-Mumford stack. If $R \rightarrow X \times_{[X/R]} X$ is an unramified monomorphism, then $[X/R]$ is an algebraic space.

All the standard geometric attributes and properties of schemes carry over to the realm of Artin algebraic stacks. In particular we can talk of a stack \mathcal{X} being locally noetherian, reduced, geometrically unbranched, regular, separated, quasi-compact, connected, irreducible etc. For example for any algebraic S -stack \mathcal{X} we have

♦ **Points of \mathcal{X} :** A point of \mathcal{X} is an equivalence class of objects

$$\xi \in \left(\coprod_{\substack{K \text{ - field} \\ S}} \text{Ob } \mathcal{X}(\text{Spec}(K)) \right) / \sim,$$

where $x_1 : \operatorname{Spec}(K_1) \rightarrow \mathcal{X}$ and $x_2 : \operatorname{Spec}(K_2) \rightarrow \mathcal{X}$ are considered equivalent if there exists a common field extension $K_1 \subset K \supset K_2$ so that $x_{1| \operatorname{Spec}(K)}$ and $x_{2| \operatorname{Spec}(K)}$ are isomorphic in $\mathcal{X}(\operatorname{Spec}(K))$.

◊ **Dimension of \mathcal{X} :** Let \mathcal{X} be locally noetherian and irreducible. Choose an irreducible atlas $X \rightarrow \mathcal{X}$ and let $R := X \times_{\mathcal{X}} X$ be the corresponding smooth equivalence relation. Define $\dim \mathcal{X} := \dim X - \dim(R/X)$. Here the relative dimension of R over X makes sense since the structure morphisms $\operatorname{pr}_i \circ \delta : R \rightarrow X$, $i = 1, 2$ are both smooth and surjective and hence $\dim(R/X) = \dim(\operatorname{pr}_1 \circ \delta) = \dim(\operatorname{pr}_2 \circ \delta) = \dim R - \dim X$. Alternatively we have $\dim \mathcal{X} = 2 \dim X - \dim R$. It can be checked [Laumon-Moret-Bailly92, Lemma 5.18] that this definition is correct and does not depend on the choice of the atlas $X \rightarrow \mathcal{X}$. Observe that according to this definition the dimension of an Artin stack can be negative. For example for a group G over a field one has $\dim BG = -\dim G$.

◊ **Sheaves on \mathcal{X} :** In order to talk about sheaves we will have to define an appropriate site first. The naive approach will be to try and define a Grothendieck topology on \mathcal{X} by taking open sub stacks as neighborhoods. However since the collection of all open sub stacks in \mathcal{X} forms a 2-category instead of a category it is clear that this naive approach cannot work directly. As usual the problem can be resolved by taking algebraic spaces as neighborhoods.

With any Artin algebraic stack one associates a site \mathcal{X}_{sm} as follows.

- The objects of \mathcal{X}_{sm} are all smooth maps $U \rightarrow \mathcal{X}$ where U is an algebraic space over S .
- The morphisms between $U \rightarrow \mathcal{X}$ and $V \rightarrow \mathcal{X}$ are diagrams of the form

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & \searrow \alpha \swarrow & \\ & \mathcal{X} & \end{array}$$

which are commutative up to a natural transformation α .

- The covering families of a smooth $U \rightarrow \mathcal{X}$ are families of morphisms

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ & \searrow \alpha_i \swarrow & \\ & \mathcal{X} & \end{array}$$

such that $U_i \rightarrow \mathcal{X}$ and $U_i \rightarrow U$ are smooth maps and $U = \cup_i \operatorname{im}(U_i)$.

If \mathcal{X} is a Deligne-Mumford stack we can define a site $\mathcal{X}_{\text{ét}}$ in exactly the same way.

Definition A.10 A presheaf of sets (abelian groups, etc.) on an Artin algebraic stack \mathcal{X} is a functor $\mathfrak{F} : \mathcal{X}_{\text{sm}}^{\text{op}} \rightarrow (\text{Set})$ ((Ab), etc.). A pre-sheaf \mathfrak{F} on \mathcal{X} is a sheaf in the smooth topology if for any smooth map $U \rightarrow \mathcal{X}$ and any covering family $\{U_i \rightarrow U\}$ in \mathcal{X}_{sm} the diagram

$$\mathfrak{F}(U) \xrightarrow{a} \prod_i \mathfrak{F}(U_i) \xrightleftharpoons[c]{b} \prod_{i,j} \mathfrak{F}(U_i \times_U U_j)$$

is exact in the sense that a is the difference kernel of b and c .

Alternatively if \mathcal{X} is presented as $[X/R]$ for some smooth equivalence relation $R \xrightleftharpoons[t]{s} X$, then a sheaf on \mathcal{X} is the same as a sheaf F on X together with an isomorphism $s^*F \cong t^*F$ satisfying the obvious cocycle condition on $X \times_{\mathcal{X}} X \times_{\mathcal{X}} X$.

Example A.11 (i) To every algebraic space Y one can associate a pre-sheaf $\text{Hom}(\bullet, Y) : \mathcal{X}_{\text{sm}}^{\text{op}} \rightarrow (\text{Set})$, $(U \rightarrow \mathcal{X}) \mapsto \text{Hom}(U, Y)$. By faithfully flat descent for algebraic spaces [Knutson71, II.3] this is a sheaf. For $Y = \mathbb{A}^1$ this sheaf is denoted by $\mathcal{O}_{\mathcal{X}_{\text{sm}}}$. Notice that by definition $\mathcal{O}_{\mathcal{X}_{\text{sm}}}(U \rightarrow \mathcal{X}) = \mathcal{O}_U$.

(ii) A sheaf of $\mathcal{O}_{\mathcal{X}_{\text{sm}}}$ -modules is a sheaf \mathfrak{F} on \mathcal{X}_{sm} such that $\mathfrak{F}(U \rightarrow \mathcal{X})$ is a $\mathcal{O}_{\mathcal{X}_{\text{sm}}}$ -module compatibly with pullbacks. A sheaf of $\mathcal{O}_{\mathcal{X}_{\text{sm}}}$ -modules is *quasi-coherent* (*coherent*) if $\mathfrak{F}(U \rightarrow \mathcal{X})$ is quasi-coherent (coherent) for all $U \rightarrow \mathcal{X}$ in \mathcal{X}_{sm} .

A.2 The truncated cotangent complex

In this section we recall, following [Laumon-Moret-Bailly92, Chapter 9], the definition of a tangent stack of an Artin algebraic stack and its relation with the cotangent complex.

Consider the functor $(\bullet)[\varepsilon] : (\text{Sch}/S) \rightarrow (\text{Sch}/S)$, defined by $U[\varepsilon] := U \times \text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$. Denote by $i : U \hookrightarrow U[\varepsilon]$ and $r : U[\varepsilon] \rightarrow U$ the canonical closed immersion and retraction respectively.

Definition A.12 The tangent groupoid of an S -groupoid \mathcal{X} is the pre-sheaf of groupoids $T_{\mathcal{X}/S}$ for which $T_{\mathcal{X}/S}(U) := \mathcal{X}(U[\varepsilon])$ and for any morphism $\varphi : V \rightarrow U$ in (Sch/S) the base-change functor $\varphi^* : T_{\mathcal{X}/S}(U) \rightarrow T_{\mathcal{X}/S}(V)$ is just the functor $(\varphi[\varepsilon])^* : \mathcal{X}(U[\varepsilon]) \rightarrow \mathcal{X}(V[\varepsilon])$.

It is not hard to check [Laumon-Moret-Bailly92, Lemma 9.13] that for an Artin stack \mathcal{X} the tangent groupoid will also be an Artin stack. It is equipped with a canonical structure morphism $\text{pr}_{\mathcal{X}} : T_{\mathcal{X}/S} \rightarrow \mathcal{X}$ defined by $\text{pr}_{\mathcal{X}}(\xi) = i^*\xi$ and with a vertex morphism $\text{ver}_{\mathcal{X}} : \mathcal{X} \rightarrow T_{\mathcal{X}/S}$ defined by $\text{ver}_{\mathcal{X}}(x) = r^*x$. Furthermore there is a natural morphism of \mathcal{X} -stacks $\gamma : \mathbb{A}^1 \times T_{\mathcal{X}/S} \rightarrow T_{\mathcal{X}/S}$ which can be described as follows. For any $\lambda \in \mathbb{A}^1$ consider the translation action $t_{\lambda} : \text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \rightarrow \text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ induced by the morphism of rings $a + b\varepsilon \mapsto a + \lambda b\varepsilon$. For any U in (Sch/S) then define the functor $\gamma(\lambda, \bullet) : \mathcal{X}(U[\varepsilon]) \rightarrow \mathcal{X}(U[\varepsilon])$ to be the base-change functor $(\text{id}_U \times t_{\lambda})^*$. We leave it to the reader to check that γ preserves $\text{ver}_{\mathcal{X}}$ and is multiplicative up to canonical 2-morphisms and thus makes $(T_{\mathcal{X}/S}, \text{ver}_{\mathcal{X}})$ into a cone stack in the sense of [Behrend-Fantechi97, Definition 1.5].

Remark A.13 (i) If \mathcal{X} is a smooth algebraic stack presented as $[X/R]$ with X, R being smooth algebraic spaces, then $T_R \rightrightarrows T_X$ is a presentation of $T_{\mathcal{X}}$.

(ii) For a Deligne-Mumford stack \mathcal{X} we can define a sheaf of Kähler differentials $\Omega_{\mathcal{X}/S}^1$ on $\mathcal{X}_{\text{ét}}$ by setting $\Omega_{\mathcal{X}/S}^1(U \rightarrow \mathcal{X}) = \Omega_{U/S}^1$. It is clear from the definition that we have $T_{\mathcal{X}/S} = \text{Spec}(\text{Sym}^\bullet \Omega_{\mathcal{X}/S}^1)$ where $\text{Sym}^\bullet \Omega_{\mathcal{X}/S}^1$ is the symmetric algebra of $\Omega_{\mathcal{X}/S}^1$. In other words in this case $T_{\mathcal{X}/S}$ is an abelian cone stack in the sense of [Behrend-Fantechi97, Definition 1.9]. If in addition \mathcal{X} is smooth $T_{\mathcal{X}/S}$ will be a vector bundle stack.

(iii) Unfortunately, the construction in (ii) cannot be applied directly to Artin stacks since in that case the naive definition of Kähler differentials used for Deligne-Mumford stacks does not work. It turns out that for a general Artin stack \mathcal{X} the tangent stack $T_{\mathcal{X}/S}$ is again an abelian cone stack. However it is very rare for $T_{\mathcal{X}/S}$ to be a vector bundle stack even when \mathcal{X} is smooth over S . Nevertheless $T_{\mathcal{X}/S}$ admits an interpretation in terms of sheaves of differentials similar to the one in (ii).

Before we briefly explain this interpretation (see [Laumon-Moret-Bailly92, Theorem 9.20] for more details), recall the following construction. Given any algebraic S -stack \mathcal{X} and any complex of sheaves of abelian groups $E^0 \rightarrow E^1$ on \mathcal{X} one may consider the stack theoretic quotient of the translation action of E^0 on E^1 . In this way we get an S -stack $h^1/h^0(E^\bullet) := [E^1/E^0]$ having also a structure of a strictly commutative group stack over \mathcal{X} (see [SGA4, Section 1.4 of Exposé XVIII] and [Behrend-Fantechi97, Section 2] for details).

Consider now an atlas $p : X \rightarrow \mathcal{X}$ for \mathcal{X} . Due to the base change property of the Kähler differentials we get a complex $\Omega_{X/S}^1 \rightarrow \Omega_{X/\mathcal{X}}^1$ of quasi-coherent étale sheaves on X . It is straightforward to check that $h^1/h^0((\Omega_{X/S}^1 \rightarrow \Omega_{X/\mathcal{X}}^1)^\vee)$ is just the stack quotient of $T_{X/S}$ by the equivalence relation

$$T_{X/S} \times_X T_{X/\mathcal{X}} \rightrightarrows T_{X/S}.$$

Combined with (i) this now gives a canonical 1-isomorphism of the pullback $T_{X/S} \times_{\mathcal{X}} X$ of $T_{X/S}$ to X and $h^1/h^0((\Omega_{X/S}^1 \rightarrow \Omega_{X/\mathcal{X}}^1)^\vee)$.

If one wants to go one step further and obtain an intrinsic description of $T_{X/S}$ in terms of differentials on \mathcal{X} only, then one is naturally lead to using the cotangent complex of \mathcal{X} .

Review of the cotangent complex

Recall that to each morphism of schemes $f : X \rightarrow Y$ Illusie [Illusie71] associates a canonical chain complex $L_{X/Y} \in \text{Ob}(C^{[-\infty, 0]}(\mathcal{O}_{X_{fppf}}))$ called the cotangent complex of f . The complex $L_{X/Y}$ has quasi-coherent cohomology sheaves and is augmented to $\Omega_{X/Y}^1$. The construction of $L_{X/Y}$ is technical and requires a somewhat advanced simplicial machinery. Rather than recalling this elaborate construction we just list those characteristic properties of the cotangent complex that are relevant to our discussion.

CHARACTERISTIC PROPERTIES OF THE COTANGENT COMPLEX - THE CASE OF SCHEMES

• **Functoriality:** $L_{X/Y}$ exhibits the same functorial behavior as $\Omega_{X/Y}^1$. More precisely any commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

gives rise to a map of complexes $g^*L_{X/Y} \rightarrow L_{X'/Y'}$. If in addition X or Y' is flat over Y then this canonical map is a quasi-isomorphism.

Furthermore, for any morphisms of schemes $X \xrightarrow{f} Y \rightarrow S$ the natural short exact sequence of complexes $f^*L_{Y/S} \rightarrow L_{X/S} \rightarrow L_{X/Y}$ extends to a distinguished triangle in $D(\mathcal{O}_{X_{fppf}})$:

$$(A.2.13) \quad f^*L_{Y/S} \rightarrow L_{X/S} \rightarrow L_{X/Y} \rightarrow f^*L_{Y/S}[1]$$

which depends functorially on $X \rightarrow Y \rightarrow S$. We will denote degree one map $L_{X/Y} \rightarrow f^*L_{Y/S}[1]$ in (A.2.13) by $e_S(X/Y)$. The element

$$e_S(X/Y) \in \text{Hom}_{D(\mathcal{O}_{X_{fppf}})}(L_{X/Y}, f^*L_{Y/S}[1]) =: \text{Ext}_{\mathcal{O}_X}^1(L_{X/Y}, f^*L_{Y/S})$$

is called *the Kodaira-Spencer class* of the morphism f . In the case when f is smooth $e_S(X/Y)$ coincides with the Atiyah class we introduced after Definition 2.5.

• **Relation to the Kähler differentials:** The cotangent complex captures the local properties of f . For example if Y is noetherian and f is locally of finite type then f is smooth iff the augmentation map $L_{X/Y} \rightarrow \Omega_{X/Y}^1$ is a quasi-isomorphism and f is a l.c.i. morphism iff $L_{X/Y}$ is quasi-isomorphic to a complex of perfect amplitude one.

More generally let $f : X \rightarrow S$ be a morphism of schemes which admits a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow f & \downarrow p \\ & & S \end{array}$$

with i a closed immersion defined by an ideal I and p smooth. Then, in $D(X)$ there is a canonical isomorphism

$$\tau_{\geq -1}L_{X/S} \simeq [0 \rightarrow I/I^2 \xrightarrow{d} i^*\Omega_{Y/S}^1 \rightarrow 0]$$

where $i^*\Omega_{Y/S}^1$ is placed in degree zero.

• **Relation to deformation theory:** If $X \rightarrow S$ is a morphism of schemes, the cotangent complex $L_{X/S}$ governs the deformation theory of X over S .

Recall that a closed immersion $i : S \rightarrow \bar{S}$ of schemes is called a square zero extension of S by a quasi-coherent sheaf M if the ideal sheaf I of S in \bar{S} is isomorphic to M as a \mathcal{O}_S -module and $I^2 = 0$ in $\mathcal{O}_{\bar{S}}$.

For any square zero extension $i : S \rightarrow \bar{S}$ we have a functoriality morphism $L_S \rightarrow L_{S/\bar{S}}$ which can be composed with the truncation $L_{S/\bar{S}} \rightarrow \tau_{\geq -1} L_{S/\bar{S}}$ to give a map $e(i) : L_S \rightarrow \tau_{\geq -1} L_{S/\bar{S}} = M[1]$. The role of the cotangent complex in deformation theory is explained by the following theorem

Theorem A.14 *Let $f : X \rightarrow S$ be a morphism of schemes and let $i : S \rightarrow \bar{S}$ be a square zero extension of S by a sheaf M . Then:*

- (i) *There exists an obstruction $\omega(X/S, i) \in \text{Ext}^2(L_{X/S}, f^*M)$ whose vanishing is necessary and sufficient for the existence of a deformation $\bar{f} : \bar{X} \rightarrow \bar{S}$ of f over \bar{S} . Furthermore the obstruction class $\omega(X/S, i)$ is the cup product*

$$\omega(X/S, i) = f^*e(i) \cup e(X/S)$$

*of the class $e(i) \in \text{Ext}^1(L_S, M)$ corresponding to $i : S \rightarrow \bar{S}$ and the Kodaira-Spencer class $e(X/S) \in \text{Ext}^1(L_{X/S}, f^*L_S)$.*

- (ii) *When $\omega(X/S, i) = 0$, the set of isomorphism classes of deformations \bar{f} is an affine space under $\text{Ext}^1(L_{X/S}, f^*M)$ and the automorphism group of a fixed deformation is canonically isomorphic to $\text{Ext}^0(L_{X/S}, f^*M)$.*

Proof. [Illusie71, Chapter III] □

Laumon and Moret-Bailly extended Illusie theory of the cotangent complex to the case of algebraic stacks [Laumon-Moret-Bailly92, Section 9]. To each 1-morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of S -stacks they associate a projective system of ind-objects $L_{\mathcal{X}/\mathcal{Y}}^{\geq -n} \in \text{Ob}(D^{[-n, 1]}(\mathcal{O}_{\mathcal{X}})) \subset \text{Ob}(D^{[-\infty, 1]}(\mathcal{O}_{\mathcal{X}}))$ such that for every n the morphism $L_{\mathcal{X}/\mathcal{Y}}^{\geq -n-1} \rightarrow L_{\mathcal{X}/\mathcal{Y}}^{\geq -n}$ induces an isomorphism of the truncation $\tau_{\geq -n} L_{\mathcal{X}/\mathcal{Y}}^{\geq -n-1}$ with $L_{\mathcal{X}/\mathcal{Y}}^{\geq -n}$. The cotangent complex $L_{\mathcal{X}/\mathcal{Y}}$ of f is defined as the projective limit of $\{L_{\mathcal{X}/\mathcal{Y}}^{\geq -n-1}\}_{n \geq 0}$ and enjoys properties paralleling the ones the cotangent complex for schemes has.

CHARACTERISTIC PROPERTIES OF THE COTANGENT COMPLEX - THE CASE OF STACKS

◇ **Normalization:** Whenever \mathcal{X} and \mathcal{Y} are representable by algebraic spaces X and Y the object $L_{\mathcal{X}/\mathcal{Y}}$ can be realized as the projective limit of the system $\{\tau_{\geq -n} L_{X/Y}\}_{n \geq 0}$.

◇ **Functoriality:** Any 2-commutative diagram of algebraic stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{X} \\ \downarrow & \searrow & \downarrow f \\ \mathcal{Y}' & \xrightarrow{\quad} & \mathcal{Y} \end{array} \quad \left(\text{or} \quad \begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{X} \\ \downarrow & \swarrow & \downarrow f \\ \mathcal{Y}' & \xrightarrow{\quad} & \mathcal{Y} \end{array} \text{ for short} \right)$$

gives rise to a morphism $\mathbb{L}g^*L_{\mathcal{X}/\mathcal{Y}} \rightarrow L_{\mathcal{X}'/\mathcal{Y}'}$. If in addition \mathcal{X} or \mathcal{Y}' is flat over \mathcal{Y} then this canonical map is an isomorphism.

Furthermore, for any morphisms of algebraic S -stacks $\mathcal{X} \xrightarrow{f} \mathcal{Y} \rightarrow \mathcal{Z}$ there exists a morphism $L_{\mathcal{X}/\mathcal{Y}} \rightarrow \mathbb{L}f^*L_{\mathcal{Y}/\mathcal{Z}}[1]$ which fits into a distinguished triangle in $D(\mathcal{O}_{\mathcal{X}_{fppf}})$:

$$(A.2.14) \quad \mathbb{L}f^*L_{\mathcal{Y}/\mathcal{Z}} \rightarrow L_{\mathcal{X}/\mathcal{Z}} \rightarrow L_{\mathcal{X}/\mathcal{Y}} \rightarrow \mathbb{L}f^*L_{\mathcal{Y}/\mathcal{Z}}[1]$$

which depends functorially on $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$. Again we get a Kodaira-Spencer class $e_S(\mathcal{X}/\mathcal{Y}) \in \text{Hom}_{D(\mathcal{O}_{\mathcal{X}_{fppf}})}(L_{\mathcal{X}/\mathcal{Y}}, (\mathbb{L}f^*L_{\mathcal{Y}/\mathcal{Z}}[1]))$ called *the Kodaira-Spencer class* of the morphism f .

◇ **Relation to the Kähler differentials:** The ind-object $L_{\mathcal{X}/\mathcal{Y}}^{\geq -0}$ in $D^{[-0,0]}(\mathcal{O}_{\mathcal{X}})$, that is - in the category of quasi coherent sheaves on \mathcal{X} , has an inductive limit which we will denote by $\Omega_{\mathcal{X}/\mathcal{Y}}^1$.

If the morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is smooth, then the projective system

$$L_{\mathcal{X}/\mathcal{Y}} = (\dots \rightarrow L_{\mathcal{X}/\mathcal{Y}}^{\geq -n-1} \rightarrow L_{\mathcal{X}/\mathcal{Y}}^{\geq -n} \rightarrow \dots \rightarrow L_{\mathcal{X}/\mathcal{Y}}^{\geq 0})$$

is essentially constant and for any atlas $p : X \rightarrow \mathcal{X}$ of \mathcal{X} there is a canonical isomorphism of ind-objects in $D^{[0,1]}(\mathcal{O}_{\mathcal{X}})$:

$$p^*L_{\mathcal{X}/\mathcal{Y}}^{\geq 0} \xrightarrow{\sim} [\Omega_{X/\mathcal{Y}}^1 \rightarrow \Omega_{X/X}^1].$$

If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a locally finitely presentable 1-morphism of algebraic stacks, then f is smooth if and only for each $n \geq 0$ the complex $L_{\mathcal{X}/\mathcal{Y}}^{\geq -n}$ is of perfect amplitude contained in $[0, 1]$.

Finally if the morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is representable and smooth the projective system $L_{\mathcal{X}/\mathcal{Y}}$ is essentially constant and is represented by the quasi-coherent sheaf $\Omega_{\mathcal{X}/\mathcal{Y}}^1$ placed at degree 0. In other words the natural augmentation morphism $L_{\mathcal{X}/\mathcal{Y}} \rightarrow \Omega_{\mathcal{X}/\mathcal{Y}}^1$ is an isomorphism.

The relation between the stacky cotangent complex and the deformation theory of the corresponding stacks carries over verbatim from the scheme case.

Finally let us remark that the tangent stack $T_{\mathcal{X}/S}$ of an algebraic stack can be reconstructed from the sheaf $\Omega_{\mathcal{X}/S}^1$ of Kähler differentials or more precisely from its background ind-object $L_{\mathcal{X}/S}^{\geq 0}$. In other words, there is (cf. [Laumon-Moret-Bailly92, 9.22.1]) a canonical 1-isomorphism of stacks

$$(A.2.15) \quad h^1/h^0((L_{\mathcal{X}/S}^{\geq 0})^\vee) \rightarrow T_{\mathcal{X}/S}$$

which when pulled back to an atlas $p : X \rightarrow \mathcal{X}$ induces the canonical isomorphism

$$h^1/h^0((\Omega_{X/S}^1 \rightarrow \Omega_{\mathcal{X}/S}^1)^\vee) \rightarrow T_{\mathcal{X}/S} \times_{\mathcal{X}} X.$$

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